

LOCATION IN SPACE

This problem comes from the *Connected Geometry* module *Habits of Mind*.

When you need to see things with your mind, it often helps to close your eyes, so that you aren't distracted by what you see with your eyes.

The following problem may not seem to be about coordinates, but it raises an important issue that using coordinates helps to solve.

1. Picture a point hanging in midair somewhere in the room. Picture a second point somewhere else.
 - a. How many different *straight lines* pass through those two points?
 - b. How many different *squares* can be built using those two points as corners?

The first of the two questions above is perfectly clear; but the second one is ambiguous—what constitutes “different” squares? The next problem illustrates the same idea.

2. How many different triangles can be built with at least one side that is 1" long? How many different squares can be built with at least one side that is 1" long?

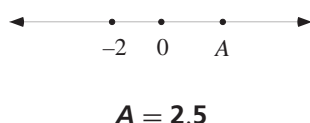
SPACES, ADDRESSES, AND DIMENSIONS

Points have no size, shape, color, weight, or flavor. They only have location.

Coordinates not only tell where things are, but also distinguish things by where they are. But in order to tell where things are, there has to be some kind of *frame of reference*. In mathematics, the “things” that coordinates locate are *points*, which cannot be distinguished *except* by their location. The points are all located, at least in the Cartesian coordinate system you'll be working with in this module, with respect to a frame of reference—a point, called the *origin*.

It helps to know something about the “space” in which the points live—whether, for example, the points are along a narrow path, or on an expansive field. In mathematics, the spaces typically have (almost) no landmarks to look at—the “paths” are lines or curves with no marks on them; the “fields” are planes or curved surfaces, again with no marks. So, with indistinguishable points in a featureless space, how do we find any point we're looking for?

A coordinate system must give every point in a space its own private address.



We need addresses (or coordinates). That's where the importance of the origin comes in. It's a landmark. The address of *every* point in the Cartesian coordinate system is specified in relation to that landmark.

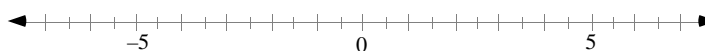
For example, a line is another space in which points live. A number line is a space with a coordinate system already built in. Every point on the number line can be located just by knowing its address. The address contains two pieces of information:

A coordinate system that uses n pieces of information to address every point in the space is called n -dimensional.

What does it mean to say “There is a one-to-one correspondence between the set of points on the line and the set of real numbers”?

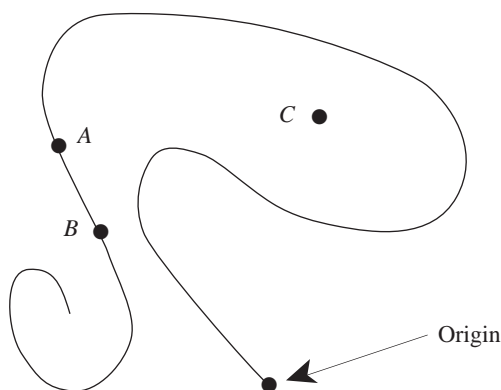
which way to go from Zero (the origin) and how far. Using negative numbers, *one* piece of information—a signed number—tells both the direction and the distance from the origin. When every point in the space can be named uniquely and conveniently using only *one* number, we call the coordinate system *one-dimensional*. The line also happens to be a *one-dimensional space*. The “address” of any point on the line—the number that corresponds to that point—is called the *coordinate* of that point.

3. On your own paper, draw a number line and mark roughly where each of these points should go.



Space 1

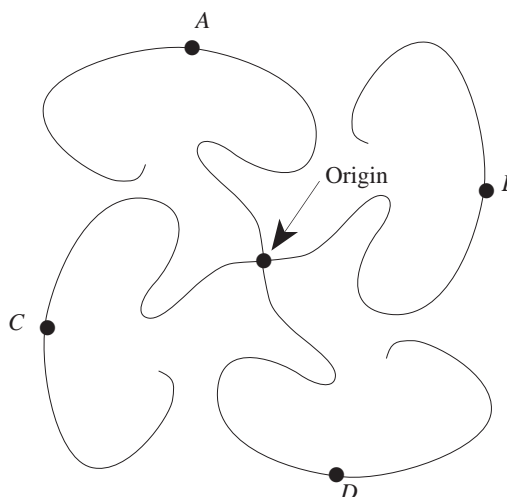
- a. $\frac{1}{2}$
 - b. -2.5
 - c. 1.2
 - d. π
4. The picture below diagrams another “narrow path.” While the number line is straight and infinite and has its origin “in the middle,” this path is twisty and finite, and has its origin at one end. Explain how you could locate any point along the path with just one piece of information.



Space 2

A , B , and the origin are not the only points in this space. But points *in this space* must be on the path. The point marked “ C ” is not a point in this space.

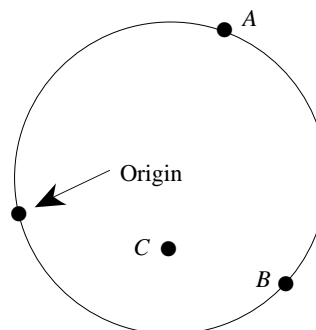
5. The diagram below is a map of another path-like “space.” This one consists of four finite paths radiating out (symmetrically) from a single point, which is treated as the origin. Invent a system of addresses—coordinates—that will identify any point in *this* space, and explain how your system works. Can you find a way of getting by with just *one* piece of information, while still being certain that every point has its own unique address, or do you need two?



Space 3

6. What about this looped-path space? The space is the circle itself, and not its interior: that is, the only points we care about addressing are *on* (not within) the circle. Invent a coordinate system that will identify each point uniquely. Will a *one-dimensional* coordinate system do?

And what about the space itself? Do you think of this as a one-dimensional space?



Space 4

It often happens that the dimensions of a space and the dimensions of a coordinate system for that space are the same. But, as an algebra teacher in Roselle, NJ was fond of saying, “Life is not always that simple.”

- 7. Write and Reflect** What *is* going on here? One day a circle is a two-dimensional figure, and the next it seems it might be used as a one-dimensional circle-space! Is *circle* being used in a different (or more specialized) way here, or is *dimension* the word that is being used differently? And what about the curved, finite path-spaces? Straight lines are one-dimensional, but curves might not be? And even then, the coordinate systems don’t seem consistent: some curves seem to require two-dimensional coordinate systems while others can use one-dimensional ones. What does “dimension” really mean?

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WAYS TO THINK ABOUT IT

Here are some things to think about when you assign addresses to the points in a space.

- On the number line, a signed number served as the single coordinate for any point. It was just an arbitrary decision that the positive numbers were placed to the right of the origin. After all, if the number line were rotated so that it went from southeast to northwest (or any other direction), the addressing system would work just as well. But someone must *indicate* which direction is “positive.” So there is *more* than just a distinguished *point*—an origin—to a coordinate system. Generally, there must also be a distinguished *direction*. Why didn’t Spaces 2 and 3 seem to need a distinguished direction? Space 4 *does* need a distinguished direction. Why?
 - So far, except for one problem, this module has not included a *scale* in the pictures of the various spaces, but you may have had to adopt a scale in answering some of the problems. For example, how far from the origin is 1? Is the measurement in inches, centimeters, or arbitrary units? Like a distinguished point and one or more distinguished directions, some kind of scale must also be specified before one can uniquely address a point.
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You could not locate $\frac{5}{2}$ on the number line if there were two places that it might go.

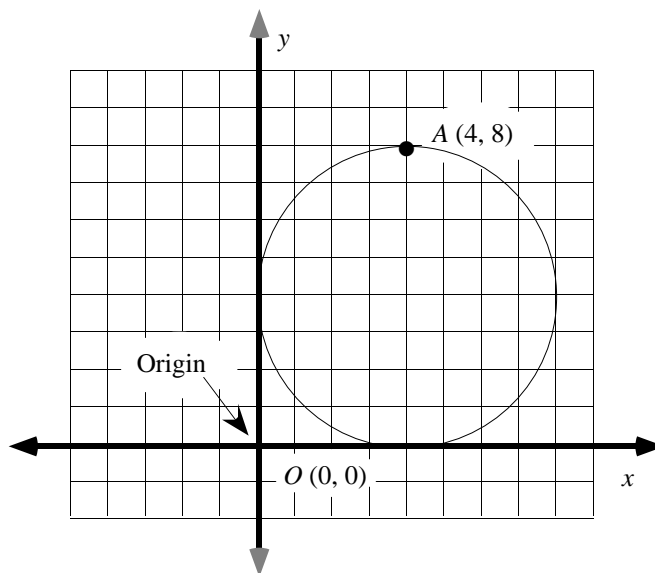
Does it bother you that $\frac{5}{2}$ and $\frac{15}{6}$ are in the same place?

FOR DISCUSSION

How useful is a labeling system in which two different points might have the same address (a coordinate or set of coordinates)?

What about the other way around? In Space 4, after you've picked an origin and a direction for "positive distance," going $\frac{1}{3}$ of the way around the circle or $\frac{4}{3}$ of the way around the circle brings you to the same point. How tolerable is that?

And how acceptable do you find an addressing system that can name points that are *not* in the space that you care about? For example, if the space is (again) a circle but the coordinate system is the one shown below, then the coordinates (100, 100) and $(-5, -2)$ name something other than points in the space.

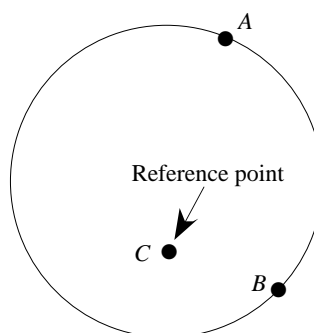


Space 5

TAKE IT FURTHER.....

While this example may seem strange, it does not violate the *purpose* of coordinates: to give unique addresses to the points in a space. One way of solving this problem is to make good use of the idea of a “distinguished direction.”

8. Space 6 is just like Space 4—points in the space are only those that lie *on* the circle and not those that lie inside or outside of it. But imagine choosing a reference point that is not part of the space itself. Let that reference point be at C . *Distance* from C will not sufficiently distinguish the points in the space (the points on the circle). Why not? Try to find a system that identifies *only* the points that you care about, and not all the other points on the plane! Can you find a one-dimensional coordinate system (as for Space 4), or does this way of looking at things seem to change the dimensions of the addressing system you need?



Space 6

LOCATING POINTS IN TWO DIMENSIONS

Even though locating a point on the spidery space in Investigation 5.1 (see Space 3 on page 3) may have seemed to require a two-dimensional coordinate system and even though the space itself requires two dimensions to *draw* it, this kind of a space itself is not typically considered two-dimensional.

Typically, a “two-dimensional space” means a *surface*—a thing that has area—like a plane or the (more-or-less spherical) surface of the Earth or a region within such a surface.

Some textbooks suggest objects like paper to represent a plane, but it's important to remember that a plane has no thickness and cannot be seen. *Everything* we can touch or see is really three dimensional, even when one or more of those dimensions is so small that it's hardly noticeable. Objects in other dimensions—one or two or four or ten—might be just as real, but we can perceive them only with our minds!

So the “lake” shown below is a two-dimensional space, and so is a page of text.



Tests often suggest objects like "paper" are illustrations of the idea of a plane, and it is important to remember that a plane has no thickness and cannot be seen. No matter how much you "squish" a line, you cannot see the point (or the line, for that matter) because they have no thickness. Everything we can touch or see is really three dimensional, even when one or more of those dimensions is so small that it is hardly noticeable. Objects in other dimensions—one or two or four or ten—might be just as real, but we can't "touch" them, and we can't "see" them.

2. Locating Points in Two Dimensions

Even though locating a point on the spidery space seemed to require a two-dimensional coordinate system, and even though the space itself required two dimensions to draw it, the spidery

Typically, "two-dimensional space" means a *surface*—a thing that has area—like a plane or the (more-or-less spherical) surface of the earth, or like a region (area) within such a surface.

The "file" shown below is such a space. So is a page of text.



A range of ten

A lake A page of text

How one chooses to coordinatize the points in a space depends partly on the nature of the space, but it also depends partly on what makes "most sense" and "greatest convenience." Think about each of the situations below, and design a two-dimensional coordinate system—a system of addresses that requires two pieces of information in each address—that makes most sense for the situation.

1. **Software Design:** To program a computer, you often need a pair of numbers that uniquely specifies the location of "characters" (letters, numbers, punctuation, spaces...) on a computer screen (or on one "window" of a computer screen). Would you use a different system if you were trying to describe the location of a letter on a printed page?

SOME THOUGHT EXPERIMENTS

How someone might choose to coordinatize the points in a space depends partly on the nature of the space, but it also depends on what makes sense and has greatest convenience. Think about one or more of the five situations that follow, and design a

two-dimensional coordinate system—a system of addresses that requires two pieces of information in each address—that you think makes most sense for the situation.

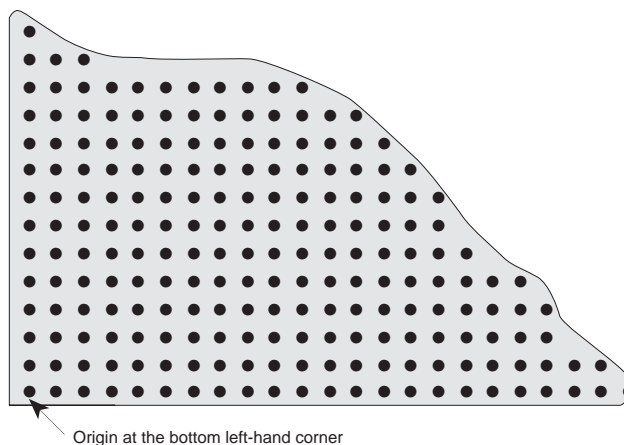
1. **Software Design** To program a computer, you often need a pair of numbers that uniquely specifies the location of “characters” (letters, numbers, punctuation, spaces, etc.) on a computer screen (or on one “window” of a computer screen). Would you use a different system if you were trying to describe the location of a letter on a printed page?
2. **Navigation or Air Traffic Control** You need to specify the exact location of a ship somewhere on a very large lake or ocean. People on the ship might have some way of figuring out their latitude and longitude and specifying their location that way, but try to find a way that they could measure from some fixed location (an “origin”) that is not on the ship. If you like, you can start by imagining that they can *see* the ship.
3. **Chess or Checkers** To record the moves in a chess game, you need names for each square on the board. Invent a system or describe the conventional one.
4. The first picture on the next page illustrates a two-dimensional array of dots. It has a bottom row, but extends infinitely upward. And it has a left-hand column, but extends infinitely to the right.
 - a. Invent a two-dimensional system of coordinates that gives each dot an address and that guarantees that any address corresponds to only one dot.

Picture the origin wherever you like: on the lake or on the shore.

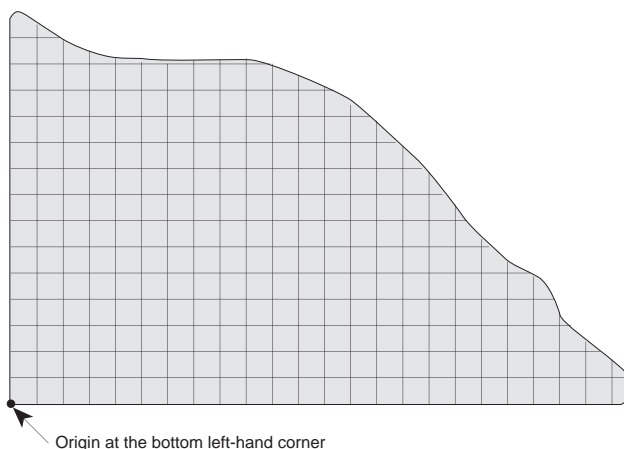
There are no dots to the left of the edge that you see, nor are there any below the bottom edge.

This challenge amounts to showing that this infinity of dots is “countable.” You can’t finish counting, of course—too many dots, too little time!—but you can show how it is possible, in theory, to assign a counting number to each dot without running out of numbers. The method also shows how the *rational numbers* are “countable.”

- b. Challenge** As amazing as it may seem, it is possible to find a scheme in which *one* integer is a perfectly adequate address for each dot: each dot is named by exactly one positive integer, and each positive integer names exactly one dot! You cannot simply number the dots in the bottom row in order, and then proceed to the next row up, because the bottom row never ends. How *could* you possibly assign the numbers 1, 2, 3, 4, . . . so that each dot gets a different number?



- 5.** The picture below also has a bottom but extends infinitely upward, and also has a left-hand side, but extends infinitely to the right. If you needed to assign coordinates only to the intersections on the grid, this problem would be the same as the one above. But this time, you must find a way to assign coordinates to any point within the entire area—*on* the lines or anywhere between them. Describe a workable scheme.



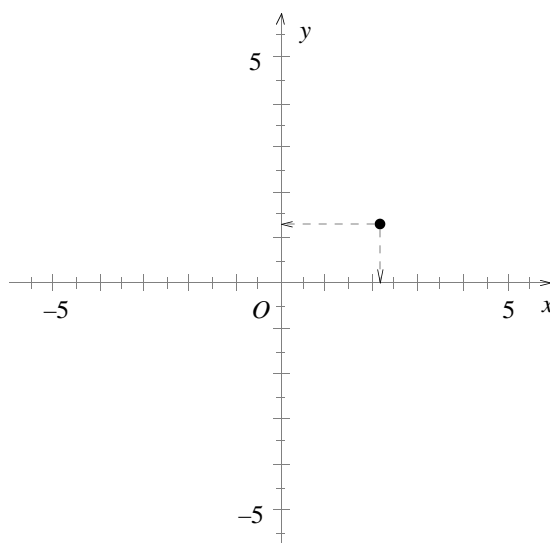
Every point within the area needs coordinates.

ONE OF TWO COMMON SYSTEMS

The two number lines are called **coordinate axes**. The point at the intersection of the axes is called the **origin**. In geometry, one often draws the axes to the **same scale** so that a circle centered at the origin passes through the same numbers on the horizontal and vertical axes.

There are two common ways to assign coordinates to points in the plane. One way uses two number lines, perpendicular to one another. Each location in the plane is described by two numbers, “measured” by the two number lines. The first number (often called the x -coordinate) comes from the horizontal number line, and says how far to the right or left of the origin the point is. The second number (often called the y -coordinate) comes from the vertical number line, and says how far above or below the origin the point is.

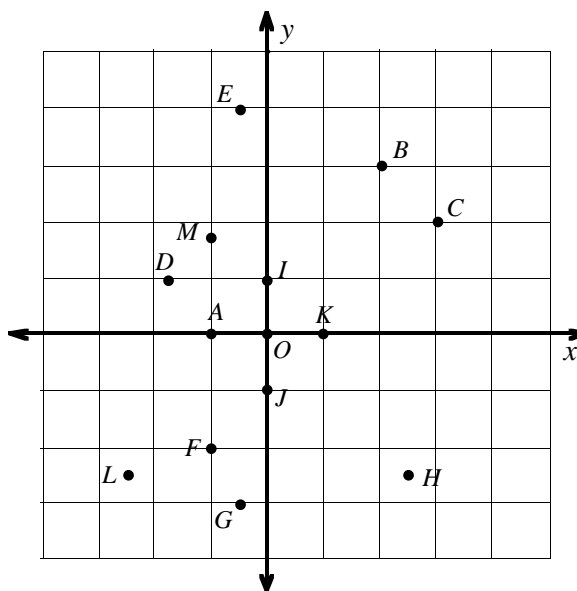
So, in this system, the x -coordinate indicates a horizontal distance, and the y -coordinate a vertical distance. The order of the two numbers distinguishes pairs like $(1, 3)$ and $(3, 1)$. This system of assigning ordered pairs of numbers to points in the plane, called the Cartesian coordinate system, is credited to René Descartes. Read the “Perspective” essay at the end of this investigation for more background on Descartes and the development of the Cartesian coordinate system.



Use a Cartesian coordinate plane in working on the following problems.

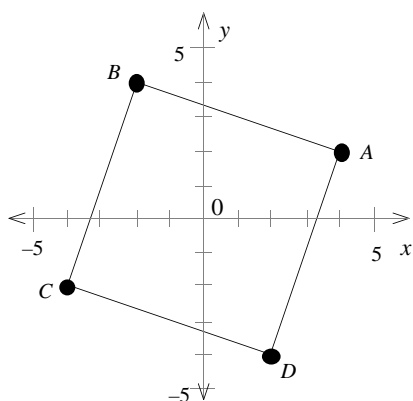
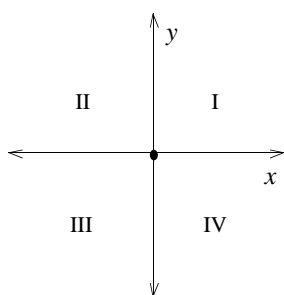
Point	Coordinates
<i>E</i>	
<i>O</i>	(,)
	(0, 1)
<i>C</i>	(3, 2)
<i>J</i>	
	(2.5, -2.5)
<i>M</i>	
	(-2.5,)
<i>D</i>	
<i>B</i>	
<i>G</i>	$(-\frac{1}{2}, -3)$
	(1, 0)
<i>A</i>	
<i>F</i>	

6. a. Complete the table at the left. Estimate the coordinates if necessary.



- b. Where would point *N*, at $(3, -3)$, be in comparison to point *H*: To the left or right? Above or below?
- c. Locate another point, *P*, *directly* above point *F*. Is the *y*-coordinate of *P* greater than, smaller than, or the same as the *y*-coordinate of *F*? How do the *x*-coordinates compare?
7. Without drawing on paper, picture a coordinate plane in your mind, and imagine locating these points:
- $(2, -1)$ and $(-2, -1)$;
 - $(4, 3)$ and $(-4, 3)$.
- How are these two pairs of points alike?
8. On a piece of graph paper, draw and label a pair of coordinate axes. Graph the points associated with the following coordinates:
- A*: $(\pi, 4)$
- B*: $(\frac{3}{4}, -5)$
- C*: $(-2, \sqrt{2})$

The x - and y -axes divide the plane into *cuatro* (Spanish for “four”) quadrants.



9. Here is a sequence of coordinates: $(1, 0)$, $(1, 1)$, $(3, 1)$, $(3, 0)$, $(4, 0)$, $(4, 1)$, $(6, 1)$, $(6, 0)$, $(7, 0)$. Extend this sequence in both directions by giving the two coordinate pairs that follow and the two coordinate pairs that precede this part of the sequence.
10. On a piece of graph paper, draw in the x - and y -axes. Pick two positive numbers, x and y , choosing values between 2 and 12. Plot the point with coordinates (x, y) , and label it A . A will be in the first quadrant of the coordinate plane.
 - a. Now graph point $(-x, y)$ on your paper and label it B . (B is the reflection of A over the y -axis.) In which quadrant did you graph B ?
 - b. Now graph point C with coordinates $(-x, -y)$. Which quadrant contains C ?
 - c. Graph $(x, -y)$ and label it D . Which quadrant contains D ?
 - d. Create the figure $ABCD$ by connecting the points. What shape do they make? Explain your answer.
 - e. Compare your shape to those drawn by classmates. What do the shapes have in common? Are they similar? Congruent? Completely different? Explain.
11. Here is a picture of a “nonlevel square” with its center at the origin.
 - a. Find the coordinates of its four vertices.
 - b. A square has vertices at $(-3, 1)$ and $(1, 3)$ and is centered at the origin. Find the coordinates of its two other vertices.
 - c. Create your own nonlevel square centered at the origin. (Make yours different from the two above.) List the coordinates of your new square.
 - d. Describe the pattern in the coordinates of the vertices of such squares.
12. a. Complete the following table.

A	B	C	D	E	F	G
(x, y)	$(x+3, y)$	$(-x, y)$	$(x, -y)$	$(2x, 2y)$	$(\frac{x}{2}, \frac{y}{2})$	$(-y, x)$
$(2, 1)$					$(1, \frac{1}{2})$	
$(-4, 0)$	$(-1, 0)$					
$(-5, 4)$		$(5, 4)$				$(-4, -5)$

Be sure to list any conjectures that you come up with.

- b.** On a piece of graph paper, plot the three points in column A and connect them to form a triangle. Plot the three points in column B, and connect *them* to form a second triangle. Describe how the two triangles differ.
 - c.** On a fresh piece of graph paper, draw triangles A and C, and describe how they differ. Repeat for triangles A and D.
 - d.** Using fresh graph paper each time, pair triangle A with each of triangles E, F, and G. Each time describe how the two triangles differ.
- 13.**
- a.** How can you tell, by looking at the coordinates, if a point is in quadrant I?
 - b.** How can you tell, by looking at the coordinates, if a point is in quadrant II?
 - c.** How can you tell, by looking at the coordinates, if a point is in quadrant III?
 - d.** How can you tell, by looking at the coordinates, if a point is in quadrant IV?

FOR DISCUSSION

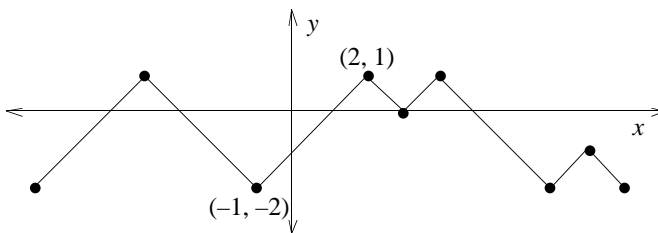
- Describe how a point moves if the sign of its x -coordinate is changed. Describe how a point moves when you change the sign of its y -coordinate. What happens when you change the sign of both coordinates?
 - Explain this statement and then decide whether you think it is true or false: “Every ordered pair of real numbers (x, y) corresponds to a unique point in the Cartesian plane; that is, there is a one-to-one correspondence between the set of points in the plane and the set of ordered pairs of real numbers.”
-
- 14.**
- a.** True or false: If point E is directly to the right of point F , then the x -coordinate of E will be greater than the x -coordinate of F .
 - b.** If E is directly to the right of F , what can you say about the y -coordinates of the two points?
 - c.** How do the y -coordinates compare when E is directly below F ? How do the x -coordinates compare?

CHECKPOINT.....

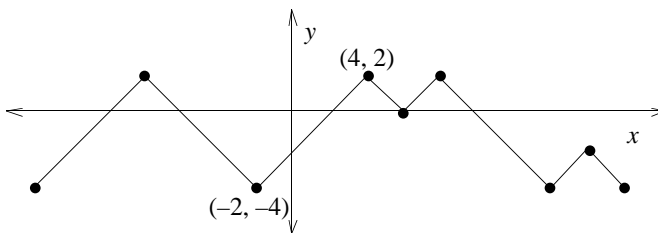
Estimate all coordinates to the nearest integer. Note which points may have x - or y -coordinates in common.

- 15.** Each of the three figures that follow is drawn on a pair of coordinate axes. The horizontal and vertical axes use the same unit of measure. The coordinates of at least one point in each figure are given to you. Make an ordered list of coordinates that might reasonably describe the points in the figure.

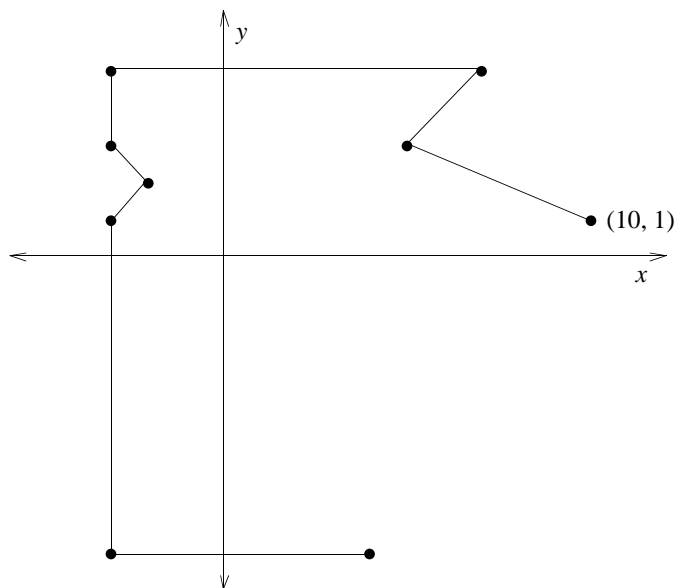
a.



b.



c.



- 16.** Compare the x -coordinates and the y -coordinates of two points C and D if:
- a.** D is directly to the left of C ;
 - b.** D is directly above C ;
 - c.** D is to the right and below C .
- 17.** On a piece of graph paper, draw three different sets of coordinate axes.
- a.** On your first set of axes, shade the quadrant(s) where the points have negative x -coordinates.
 - b.** On the second set of axes, shade the quadrant(s) where the points have positive y -coordinates.
 - c.** On the third set, shade in the quadrant(s) where the points have negative x -coordinates and a positive y -coordinates.

18. If a is a negative number and b is a positive number, state the quadrant in which each of the following points lies:

- a. (a, b)
- b. $(-a, b)$
- c. $(-a, -b)$

TAKE IT FURTHER.....

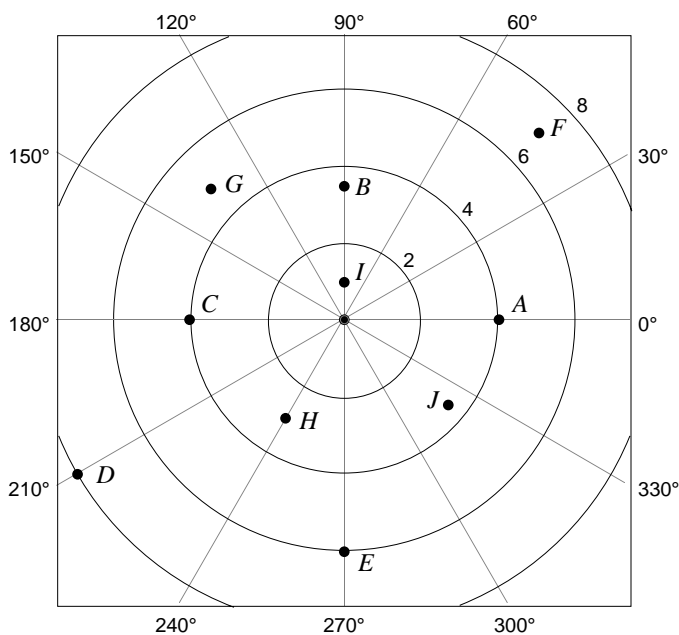
19. Each problem below describes a line in a coordinate plane. For each line, state how many quadrants it passes through.
- a. A vertical line that passes through the origin
 - b. A vertical line that does not pass through the origin
 - c. A nonvertical, nonhorizontal line that passes through the origin
 - d. A nonvertical, nonhorizontal line that does not pass through the origin
20. Is it possible for a line to pass through only one quadrant? Is it possible for a line to pass through all four quadrants? Is there *any* slanting (nonvertical, nonhorizontal) line that passes through only two quadrants and does *not* go through the origin?

This is the second system commonly used to assign coordinates to points in the plane. The point at the center of the system is called the *pole*. And the system is called *polar coordinates*.

Another way to locate a point on a plane is to describe how far it is from some central location, and in which direction. The distance might be reported in millimeters or miles (or arbitrary units), and the direction might be reported in degrees (or some other appropriate unit).

21. a. Use this distance-direction scheme to complete the table in the margin. Estimate the coordinates if necessary.

Point	Coordinates
E	
	(3, 240°)
C	(4, 180°)
	(, 320°)
	(8,)
B	($3\frac{1}{2}$,)
G	(, 135°)
	(1, 90°)
A	
	(7,)



- b. In a coordinate scheme like this, how many points can have coordinates $(4, a^\circ)$? Describe the location in which such points are found.
- c. How many points can have coordinates $(r, 60^\circ)$? Describe the location in which such points are found.
- d. How many points can have coordinates $(0, a^\circ)$?

Because the plane is two-dimensional, any coordinate system must name each point by a *pair* of numbers. The numbers in the pair mean different things. In the polar coordinate system, one number indicates a distance (in some unit) from the pole, and the other indicates direction in degrees.

PERSPECTIVE ON DESCARTES

DESCRIBING GEOMETRIC IDEAS WITH ALGEBRAIC LANGUAGE

How long have people been using coordinate systems? Who was Descartes, and why is he famous? This essay will help you answer these questions.

How many years were there between 500 B.C. and 1637 A.D.? Between 1637 A.D. and now?

The idea of using grid systems to help specify locations is a very old one. In the West, the use of grids in map making apparently dates back to Ptolemy, roughly the 2nd century A.D. But Western map making declined dramatically after Ptolemy and was relatively primitive by the 13th to 14th centuries. In China, the use of coordinates in map making began equally early, but that use continued to be refined and developed through the centuries and far surpassed Western work by the Middle Ages.

Even though the idea of coordinates developed early, its connection with geometric reasoning is relatively new in the history of Western mathematics. Euclid systematized geometric ideas that dated back at least as far as the 5th century B.C., but all that geometric thinking was done without coordinates. The systematic use of coordinates is usually credited to René Descartes around 1637—a rediscovery of an old idea or perhaps a new connection between two old ideas.

THE LIFE AND WORK OF DESCARTES

In 1637, the great French philosopher, scientist, and mathematician René Descartes (1596–1650) published a famous book called *La Géométrie*, in which he first showed how to use coordinates to translate geometry problems into algebra and vice versa. (About the same time, another Frenchman, Pierre de Fermat, independently had similar thoughts, but his ideas weren't published until 1679.) In honor of Descartes, we often call coordinate geometry *Cartesian geometry*, and we call a coordinate system of axes a *Cartesian coordinate system*.

Actually, the word “Cartesian” comes from the Latin version of René Descartes' name: Renatus Cartesius. In the 17th century, it was customary for scholars and scientists to take Latin names. The Mercator projection is another interesting example of the use of a Latin name. The Mercator projection is the most popular kind of world map and is named in honor of Mercator, whose real name was Kremer (Mercator and Kremer are the Latin and German versions of the word *merchant*).

Why x and y ? There's really nothing special about those names. When you are graphing the distance traveled in a given time, the best labels for the axes might be d and t . But "horizontal" and "vertical" don't quite work, either: After all, those words depend on how you hold the paper! The words *abscissa* and *ordinate* are hardly known anymore, let alone used, but you might enjoy reviving them (after you look them up).

If you took a look at Descartes' book *La Géométrie*, it would look quite unfamiliar. First, Descartes generally uses oblique axes; that is, his x - and y -axes do not meet at a right angle. Second, Descartes uses only positive numbers, which means that he does all his geometry in what we now call the "first quadrant." This is because in Descartes' time people did not believe that there was such a thing as a negative number. (After all, how could there be anything less than nothing?)

Descartes also intentionally made his book difficult to read. The reasons for this are quite interesting. Descartes tells us that he deliberately left out a lot of explanations so that other people could not claim that he had "written nothing that they did not already know"! He also said that he didn't want "to deprive his readers of the pleasure of working things out for themselves."

Descartes made many important contributions to mathematics, but he was also an eminent scientist, and he is perhaps best known as a philosopher. In his philosophical writings he was particularly interested in the question of how we can know for certain that the things we believe are really true. His philosophical writings are still studied and admired today; you may someday have the pleasure of reading them for yourselves. The most famous quotation from Descartes is the Latin sentence "*Cogito, ergo sum*," which means "I think, therefore I am."

Although Descartes was a Frenchman, he spent most of his adult life in Holland, where he was living when *La Géométrie* was published. You may wonder why he chose to live in a foreign country. Some of Descartes' ideas in science and philosophy were extremely controversial. In most European countries at that time, there was strict censorship of what could be published, and having unpopular ideas could be very dangerous. The notable exception to this kind of intolerance was Holland, so Descartes went there to live so that he could say, write, and publish whatever he pleased.

Descartes was not the only one who went to Holland to enjoy freedom of thought. When the Pilgrims left England in 1608 because of religious persecution, they first went to Holland. They lived there until 1620, when they left to make a new start in America. Descartes arrived in Holland in 1629, so he and the Pilgrims missed each other by just a few years.

Descartes' contribution—the idea of describing points in terms of coordinates—opened up the possibility of describing geometric ideas in algebraic language. Objects like lines, curves, and surfaces; relationships like perpendicularity; transformations like translation, rotation, and dilation all could be expressed in terms of functions and equations. This blending of algebra and geometry is called *coordinate* or *analytic geometry*.

LINES, MIDPOINTS, AND DISTANCE

From working on the problems in this investigation, you will extend your knowledge about how geometric ideas—horizontal and vertical lines, collinearity, intersection, perpendicularity, midpoints, and distance—are reflected in the coordinates of points on a Cartesian plane whose axes have a 1–1 scale.

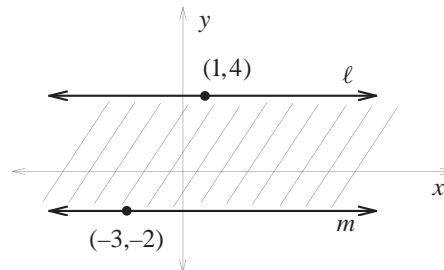
COORDINATE PRACTICE

Graph ℓ if that's helpful.
While there are infinitely many lines through it at any one point, there can be only one passing through it in any given direction. In particular, only one is vertical, and only one is horizontal.

Visualize this or draw a rough sketch without using graph paper.

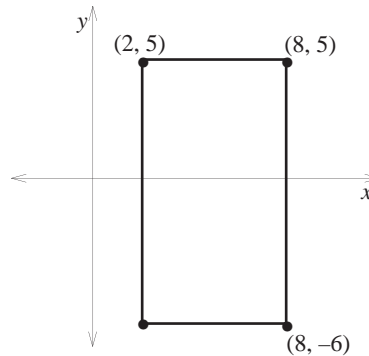
1. The y -axis is the only vertical line that can pass through the point $(0, 0)$. What do the coordinates of all the points on that line have in common? (How can you tell if a point is on the y -axis just by looking at its coordinates?)
2. Picture a horizontal line through the point $(1, 1)$ in a coordinate plane. Name two other points on that line.
3. Suppose ℓ is the vertical line passing through $(3, 7)$.
 - a. Find the coordinates of four points that are on ℓ .
 - b. Find the coordinates of four points that are not on ℓ .
 - c. How can you tell if a point is on ℓ just by looking at its coordinates?
 - d. Draw a different vertical line on a coordinate plane. What do all of the points on your line have in common?
4. Suppose m is the horizontal line passing through $(3, 7)$.
 - a. Find the coordinates of four points that are on m .
 - b. Find the coordinates of four points that are not on m .
 - c. How can you tell if a point is on m by looking at its coordinates?
5. Let A and B be two different points on the same horizontal line. If $A = (s, t)$, what coordinate of B do you know?
6. What are the coordinates for the intersection of the horizontal line through $(5, 2)$ and the vertical line through $(-4, 3)$?

7. **a.** Name and plot four points whose first coordinate is the same as the second coordinate.
- b.** Imagine a drawing that shows every point whose first coordinate is the same as the second coordinate. What shape would that be?
8. **a.** Name and plot four points whose first coordinate is the negative of the second coordinate.
- b.** Imagine a drawing that shows every point whose first coordinate is the negative of the second. What shape would that be?
9. Line a is a horizontal line through $(-5, 12)$. Line b passes through the origin and makes a 45° angle with the axes as it enters quadrant I. Find the coordinates of the point where these two lines intersect.
10. In this picture, ℓ and m are horizontal lines.



- a.** Find the coordinates of four points between ℓ and m .
- b.** Find the coordinates of four points that aren't between ℓ and m .
- c.** How can you tell if a point is between ℓ and m just by looking at its coordinates?

11. Here's a rectangle:



- Find the coordinates of four points that are inside the rectangle.
 - Find the coordinates of four points that are outside the rectangle.
 - Find the coordinates of four points that are *on* the rectangle.
 - How can you tell if a point is inside the rectangle just by looking at its coordinates?
12. Draw a set of x - and y -axes.
- Shade in the region where the points have x -coordinates such that $5 \leq x \leq 8$.
 - Now shade in the region where the points all have y -coordinates such that $-3 \leq y \leq 6$.
 - Is there any overlap in the shaded regions? If so, what shape is the intersection? What is its area in square units?
 - How can you change the instructions above to make the shaded intersection take the shape of a square? What is the area of your square?
13. One side of a quadrilateral passes through $(4, 6)$ and $(4, 3)$. Another side passes through points $(-2, 5)$ and $(-1, 2)$. Which of the following statements are true?
- The figure could not be a trapezoid.
 - The figure could be a trapezoid.
 - The figure must be a trapezoid.
 - The figure could not be a parallelogram.

- e. The figure could be a parallelogram.
- f. The figure must be a parallelogram.

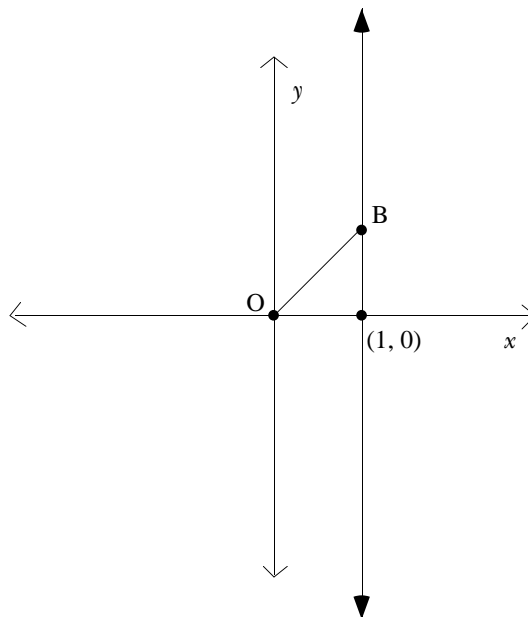
MIDPOINTS AND DISTANCE BETWEEN POINTS

Notation: An easier way to write “ A is at $(5, 7)$ ” is “ $A = (5, 7)$.”

- 14. Suppose A is at $(9, 5)$, and B is at $(7, 5)$.
 - a. What is the distance between A and B ?
 - b. Find the coordinates of the midpoint of \overline{AB} .
- 15. Suppose $C = (2, 5)$ and $D = (2, 396)$.
 - a. Find the distance between C and D .
 - b. Find the coordinates of the midpoint of \overline{CD} .
- 16. $E = (-5, -7)$, and $F = (12, -7)$.
 - a. What is the distance between E and F ?
 - b. Find the coordinates of the midpoint of \overline{EF} .
 - c. State a conjecture about the coordinates of midpoints.
- 17. Find the distance between the given pair of points:
 - a. $I = (-110, -7)$ and $J = (-80, -7)$
 - b. $K = (1, 5)$ and $L = (1, -15)$
 - c. $M = (-93, 4)$ and $N = (90, 4)$
- 18. Here are the coordinates of four points:
 - $A = (4, 2)$
 - $B = (8, 5)$
 - $C = (-4, 3)$
 - $D = (-7, 7)$.

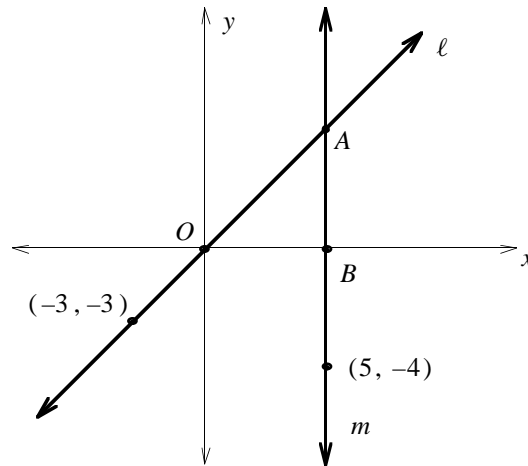
Which of the following statements can you justify? How?

- a. \overline{AB} and \overline{CD} are definitely congruent.
 - b. \overline{AB} and \overline{CD} are approximately the same length, but are not definitely congruent.
19. Find the distances between the points:
- a. $P = (a, b)$ and $Q = (a, c)$.
 - b. $P = (a, c)$ and $Q = (b, c)$.
20. What about the distance from O to B where $O = (0, 0)$ and $B = (1, 1)$? Solve the problem any way you can.



21. The following are the coordinates for the three vertices of a triangle:
 $E = (-1, -3)$, $F = (2, -3)$, and $G = (2, 1)$.
- a. How long are \overline{EF} and \overline{FG} ?
 - b. Find the distance EG .

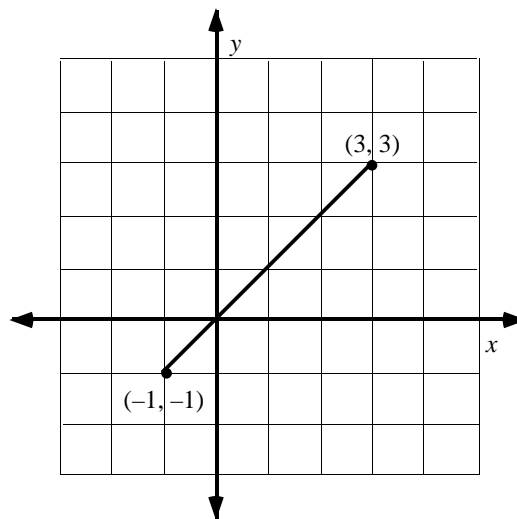
22. In this picture m is a vertical line:



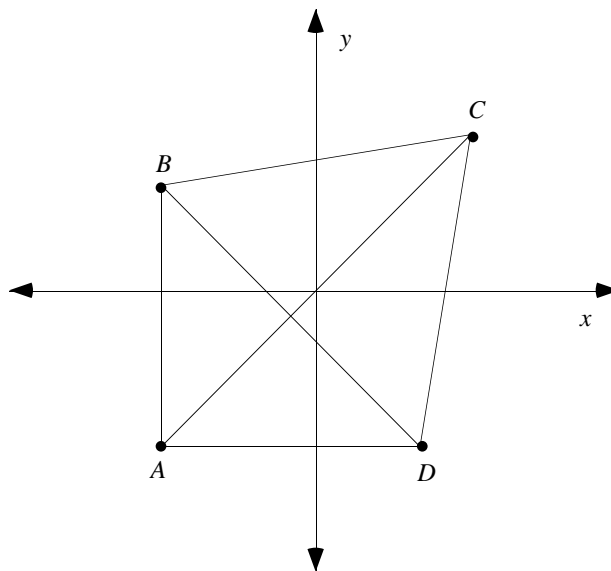
If you aren't familiar with the dimensions of the particular triangles in Problems 21 and 22, use the Pythagorean Theorem.

- a. Find the coordinates of A and B .
 - b. How long is \overline{AB} ?
 - c. What is the area of $\triangle AOB$?
 - d. How long is \overline{AO} ?
23. Use one set of axes for the following:
- a. Find eight points that are 5 units away from the origin.
 - b. Draw the picture of all the points that are 5 units away from the origin. What shape is it? Why?
 - c. Find eight more points on the figure you found in (b).
24. **Write and Reflect** Write a set of instructions that explains how to find the distance between *any* two points if you know their coordinates.
25. **Write and Reflect** Exchange your instructions with a partner. See if you can follow your partner's instructions to find the distance between points at $(1, 3)$ and $(6, 15)$.

26. Devise a method to find the coordinates of the midpoint of the segment below.



27. In the picture below $A = (-3, -3)$, $B = (-3, 2)$, $C = (3, 3)$, and $D = (2, -3)$. Use your method from Problem 26 to find the coordinates of the midpoints of \overline{AB} , \overline{CD} , \overline{AD} , \overline{BC} , \overline{AC} , and \overline{BD} . If the method you used for Problem 26 doesn't work here, look at Problem 26 again and try to develop a method that will work here.



Or write a Logo procedure that accepts (as lists) the coordinates for any two points, and returns the coordinates for the midpoint of the segment between them.

- 28. Write and Reflect** There are many correct ways to find the coordinates of the midpoint of a line segment when you know the coordinates of the endpoints. Describe your method precisely, and explain why it works.
- 29.** Assume that $G = (x, y)$ and $H = (w, z)$. Find the coordinates of the midpoint of \overline{GH} . The goal of this problem is for you to translate your instructions from Problem 28 into a mathematical formula.
- 30.** Exchange the methods you produced in Problems 28 and 29 with a partner. See if you can successfully follow your partner's methods in finding the coordinates of the midpoint of \overline{JK} , where $J = (-2, 1)$ and $K = (1, 6)$.

CHECKPOINT.....

Do these problems without using any graph paper.

- 31. a.** How many vertical lines can be drawn through the point $(-2, 3)$?
- b.** Name the coordinates of the intersection of a horizontal line through $(3, -5)$ and a vertical line through $(-1, 9)$.
- 32.** Picture (or draw) line ℓ through $(6, 4)$ and $(-3, 1)$.
- a.** How many lines are parallel to ℓ ?
- b.** Line m passes through $(7, 4)$ and is parallel to ℓ . Name a point on line m other than $(7, 4)$.
- c.** Line n passes through $(8, 3)$ and is parallel to ℓ . Name a point on line n other than $(8, 3)$.
- 33. a.** Name four other points on the horizontal line through $(3, 4)$.
- b.** Name three points that are collinear with (on the same line as) $(4, 2)$ and $(4, -3)$.
- 34.** Segments \overline{AB} and \overline{CD} bisect each other. Find E , the point of bisection, and D , if $A = (110, 15)$, $B = (116, 23)$, and $C = (110, 23)$.

Picture a set of axes in your mind's eye or make a rough sketch on plain paper; locate the point in question, and visualize the line.

Two segments bisect each other if they intersect at each other's midpoint.

- 35.** A segment has a length of 25 units. Give possible coordinates for the endpoints of this segment if it is:
- a.** on a horizontal line;
 - b.** on a vertical line;
 - c.** neither a horizontal nor a vertical segment.
- 36.** A segment has its midpoint at $(8, 10)$. List four possibilities for the coordinates of its endpoints.
- 37.** A segment has one endpoint at $(-7, -2)$, and its midpoint at $(-2, 1.5)$. What are the coordinates of the other endpoint?
- 38.** Consider point P at $(5, 0)$ and Q at $(15, 0)$.
- a.** Find six points that are just as far from P as they are from Q .
 - b.** Find six points that are closer to P than they are to Q .
 - c.** How can you tell if a point is equidistant from P and Q just by looking at its coordinates?

TAKE IT FURTHER.....

- 39.** Suppose $A = (4, 1)$, $B = (8, -2)$, $C = (7, 1)$, and $D = (3, 4)$.
- a.** Show that the diagonals of $ABCD$ bisect each other.
 - b.** What does this say about the kind of quadrilateral $ABCD$ is? Why?
- 40.** Line a passes through $(0, 1)$ and $(1, 0)$. Line b passes through the origin and makes a 45° angle with the axes as it enters quadrant I. Find the coordinates where these two lines intersect.

FORMULAS FOR MIDPOINT AND DISTANCE

In this investigation, you will formalize the instructions you wrote for finding the midpoint of and the distance between two points. You will begin by analyzing the descriptions of two different methods for finding a midpoint. Then, using subscripts, you will translate from the description of a method to the formal language of mathematics, with enough precision to prove a theorem.

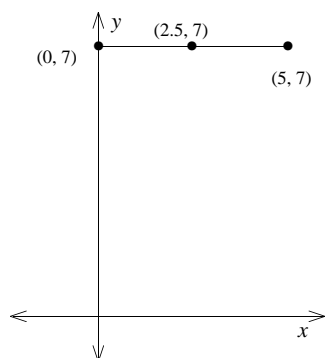
MIDPOINTS

1. First, use your own method to find the midpoints of the segments whose endpoints are listed below. Then, read the dialogue below, in which two students discuss *their* methods.

a. $(0, 7)$ and $(5, 7)$

b. $(-3, -3)$ and $(-1, 2)$

“Field tests” are classroom trials in which new materials are tested to see if students understand and like the way the mathematics is presented.



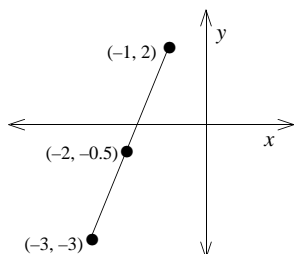
The following is adapted from a discussion that took place at one of the field test sites for the *Connected Geometry* modules. Read the dialogue, and try to understand both perspectives.

Kesia: Look, to find a number halfway between two others, you just find their average: add and divide by 2. So, to get a *point* halfway between two others, just average the x -coordinates and average the y -coordinates. So, for $(0, 7)$ and $(5, 7)$, the average of 0 and 5 is 2.5—the 7 just stays, of course. It works in three dimensions, too, and probably in any dimension.

Paul: I look at the x - and y -coordinates separately, too. But instead of averaging, I just see how far apart the points are and split the difference. The difference between the two x -coordinates gives me the horizontal distance [between the points]. Then I get the vertical distance. I cut those two distances in half and use the results to find a point halfway between.

Part a is only horizontal distance. Five minus zero is five; divide that in half to get 2.5; add that back to the zero. The midpoint is $(2.5, 7)$.

In part b, there's more to do. The horizontal distance is 2; half of 2 is 1. So I start with the smaller value, -3 , and count up 1 to get -2 .



I do the same thing for the y values. The vertical distance is 5. Half of that is 2.5, and 2.5 plus -3 is -0.5 , so the midpoint is $(-2, -0.5)$.

Kesia: I guess either method works, 'cause I get the same answer. The average of -3 and -1 is -2 , and the average of -3 and 2 is -0.5 . Ta, da!

FOR DISCUSSION

Does Paul's method always work? Does *Kesia's*? Explain.

What do the following expressions have to do with the discussion?

$$a + \frac{b - a}{2} \qquad \frac{a + b}{2}$$

2. Write a formula or algorithm for each method, *Kesia's* and Paul's.
3. Use each method to find the midpoint A between $(-1, -3)$ and $(4, 5)$, and the midpoint B between $(-16, 39)$ and $(87, 116)$. Do the two methods produce the same results? Are the two methods equally easy?

SUBSCRIPTS

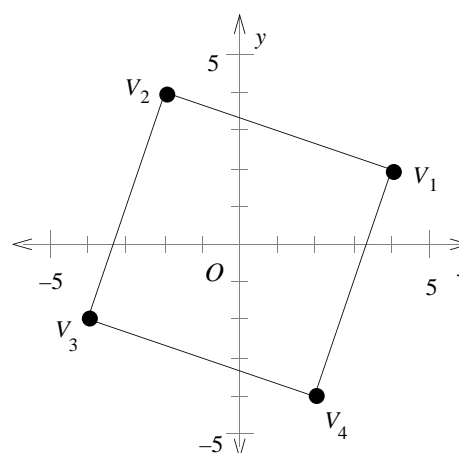
Subscript notation is used in the standard formulas for midpoints and distance, so it is introduced here. If you are already quite familiar with subscript notation, you might want to do a few of these problems and then skip to Problem 12.

When only a few points need names, it is convenient enough to call them A , B , C , and so on, with coordinates (a, b) , (c, d) , (e, f) , and so on. But it is often important to have names for many points, and then one quickly runs out of letters. Numbers *never* run out, and so the convention is to name, say, the vertices of a decagon with names like these A_1 , A_2 , A_3 , \dots , A_{10} , and the vertices of an n -gon with names like these:

$B_1, B_2, B_3, \dots, B_n$. The notation is supposed to “make sense.” See if you can figure out the logic of the notation in the problems below.

4. Look at the drawing of the square, and complete the table below. Assume that point V_1 has coordinates (x_1, y_1) and point V_2 has coordinates (x_2, y_2) , and so on.

i	Coordinates of V_i	x_i	y_i
1	(,)		
2	(,)	-2	
3	$(-4, -2)$		
4	(,)	2	



Square $V_1V_2V_3V_4$

5. Here is a claim about the coordinates of the vertices of square $V_1V_2V_3V_4$:
- $$x_i = y_{i+1}.$$
- Is that claim true when $i = 1$? That is, is it true that $x_1 = y_2$?
 - Is that claim true when $i = 2$?
 - Is that claim true when $i = 3$?
 - Is that claim true when $i = 4$?
6. Here is another claim about the vertices of square $V_1V_2V_3V_4$: “If V_i has coordinates (x_i, y_i) , then V_{i+1} has coordinates $(-y_i, x_i)$.”

a. When $i = 2$, that statement says: “If V_2 has coordinates (x_2, y_2) , then V_3 has coordinates $(-y_2, x_2)$.” Look at the table and decide whether this is a true statement.

b. Pick a value of i for which the statement doesn’t make sense.

7. Name the vertices of the square for which it is true that $y_i = \frac{1}{2}x_i$.

Finally, some new points!

$$P_3 = (y_2, -x_2)$$

8. Start with point $P_1 = (3, 4)$ and find the coordinates of P_2 , P_3 , and P_4 , if the points follow the rule that “If P_i has coordinates (x_i, y_i) , then P_{i+1} has coordinates $(y_i, -x_i)$.” Plot and label the points.

9. Here is a rule for deriving a new set of points Q_i from the points V_1 , V_2 , V_3 , and V_4 : “If $V_i = (x_i, y_i)$, then $Q_i = (-3 + x_i, 4 + y_i)$.”

a. The rule is written in algebraic symbols. Explain it in words.

b. Find the new points Q_1 , Q_2 , Q_3 , and Q_4 , and plot them.

10. If $P_1 = (x_1, y_1)$ and you know that P_2 is a second point on the same horizontal line, how could you write its coordinates?

11. $P_i = (x_i, y_i)$, $x_i = i + 3$, and $y_i = x_i - 4$. Plot P_i as i goes from 1 to 8.

BACK TO MIDPOINTS

Both Paul and Kesia have methods that work for finding a midpoint. Kesia’s can be written as a formula for M , the midpoint between two points (x_1, y_1) and (x_2, y_2) :

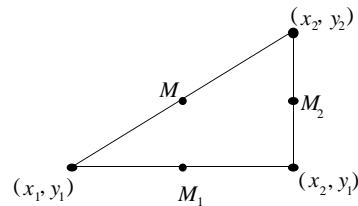
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

12. Translate the formula into a sentence in English. Because this is a method *that works* (you’ll soon see a proof), it is a theorem!

THEOREM 5.1 Midpoint Formula

Each coordinate of the midpoint of a line segment is equal to the average of the corresponding coordinates of the endpoints of the line segment.

Here's one proof of the theorem. You will have to fill in some of the details.



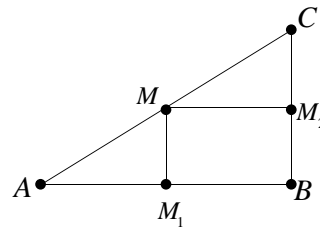
The goal is to find the coordinates of M , the midpoint of the segment connecting (x_1, y_1) and (x_2, y_2) . We'll use what we know about horizontal lines and vertical lines to find a third point, (x_2, y_1) . Then we'll find the coordinates of M_1 and M_2 .

13. Find the coordinates of M_1 .
14. Give a reason why the second coordinate of M_1 should be y_1 .
 - a. Whatever expression you used for the first coordinate of M_1 , show algebraically that
$$\text{your expression} - x_1 = x_2 - \text{your expression}.$$
 - b. Explain what the algebra was intended to prove about the midpoint.
15. Now, what are the coordinates of M_2 ? Explain how you can be sure that the coordinates you found describe a point that is not only equidistant from endpoints (x_2, y_1) and (x_2, y_2) , but also *on the line* that connects them.
16. Show that the algebraic statement

$$y_2 - \frac{y_2 + y_1}{2} = \frac{y_2 + y_1}{2} - y_1$$

is true. Explain how it fits with Paul's and Kesia's claims.

Here is a new drawing on which some of the points have new labels to make them easier to talk about.



From the drawing, it certainly looks as if we're finished! M should have the x -coordinate of M_1 and the y -coordinate of M_2 . There certainly *is* a point with those coordinates, but how can we be sure that it actually lies on \overline{AC} ?

The Side-Splitting Theorem is discussed in detail and proved in the module **A Matter of Scale**: A line parallel to one side of a triangle cuts the other two sides proportionally.

The Side-Splitting Theorem says that a line through M_1 and parallel to \overline{BC} will intersect \overline{AC} at its midpoint. Also, a line through M_2 and parallel to \overline{AB} will intersect \overline{AC} at its midpoint. Since triangle ABC is a right triangle, a vertical line through M_1 and a horizontal line through M_2 will intersect at the midpoint of \overline{AC} . That completes the proof.

17. Use the Midpoint Theorem to find the midpoint between $(1,327, 94)$ and $(-668, 17)$.
18. What's the midpoint between $(1,176, 13)$ and $(2,000, 50)$?
19. Three vertices of a square are $(-1, 5)$, $(5, 3)$, and $(3, -3)$. Find the center of the square. Find the fourth vertex.
20. Three vertices of a square are $(-114, 214)$, $(186, 114)$, and $(-214, -86)$. Find the fourth vertex and the center of the square.
21. Points A and B are endpoints of the diameter of a circle, and C is the center of the circle. Find the coordinates of C given the following coordinates for A and B .
 - a. $(-79, 687)$, $(13, 435)$
 - b. $(x, 0)$, $(5x, y)$
22. Points D and E are the endpoints of one of the sides of a square. If the coordinates of the midpoint of the side, F , are $(4.5, 17)$ and the coordinates of D are $(2, 16)$, what are the coordinates of the other endpoint, E ?

For Problems 23–25, if you take the segment and divide it into 3 or 4 or n equal parts as Paul might, that's one thing. But if you're going to add the coordinates and try to divide by 3 or whatever (as Kesia might), then you're taking the average of 3 points. In this case it's like a "weighted average": if you're going to divide by 3, you've got to *add* 3 points. The question is, *which* 3 points?

TAKE IT FURTHER.....

23. $A = (2, 1)$ and $B = (32, 1)$. Find point P that is $\frac{1}{3}$ of the way from A to B . Explain how you did it.
24. $C = (2, 2)$ and $D = (30, 2)$. Find point Q that is $\frac{1}{4}$ of the way from C to D . Explain how you did it.
25. Consider the points $E = (5, 2)$ and $F = (11, -1)$. Find point S that is $\frac{1}{3}$ of the way from E to F . Explain how you did it.
26. Let $A = (-3, 5)$, $B = (5, 1)$, and $C = (7, -9)$.
 - a. Calculate midpoint D of \overline{AB} .
 - b. Calculate midpoint E of \overline{BC} .
 - c. Calculate F to be $\frac{2}{3}$ of the way from A to E .
 - d. Calculate G to be $\frac{2}{3}$ of the way from C to D .
 - e. Calculate midpoint H of \overline{AC} .
 - f. Calculate J to be $\frac{2}{3}$ of the way from B to H .
 - g. If you have not already done so, draw the picture that goes with these calculations.

DISTANCE

It's nice to know that even if you forget the distance formula shown here, you can figure it out again just by remembering $a^2 + b^2 = c^2$.

THEOREM 5.2 The Distance Formula

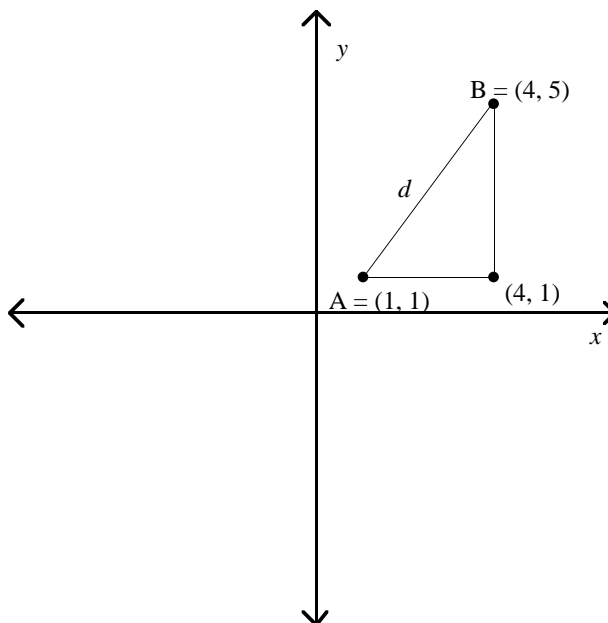
The distance d between two points (x_1, y_1) and (x_2, y_2) can be found using the Pythagorean Theorem:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

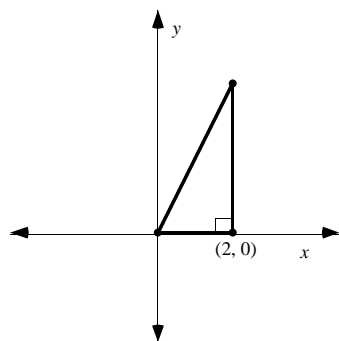
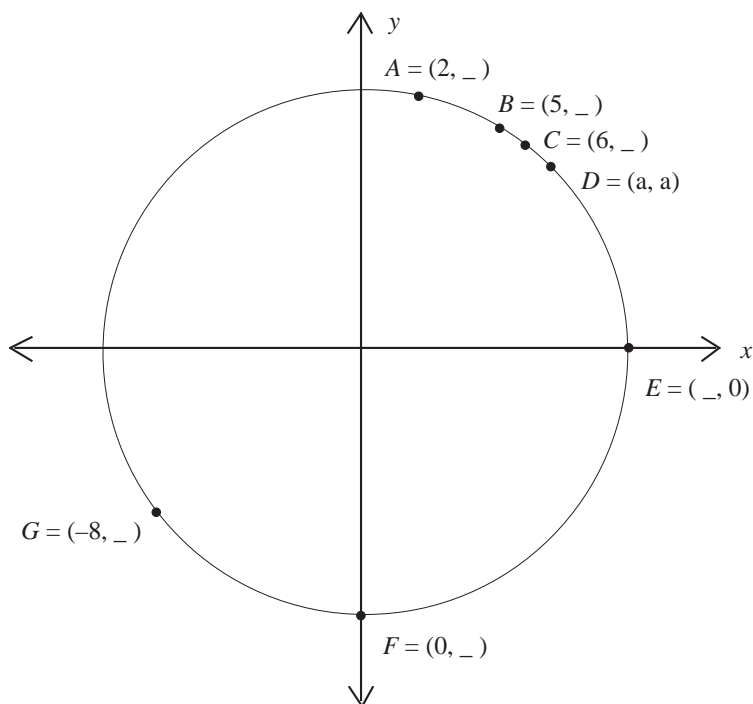
FOR DISCUSSION

The formula may not “look like” the Pythagorean Theorem, or like $a^2 + b^2 = c^2$. How *could* you explain, or rediscover, the distance formula from the Pythagorean Theorem?

If you need help getting started, use this example where $A = (x_1, y_1)$, $B = (x_2, y_2)$, and d = the distance between them.



- 27.** Find the missing coordinates for points A through G as shown in the circle of radius 10.

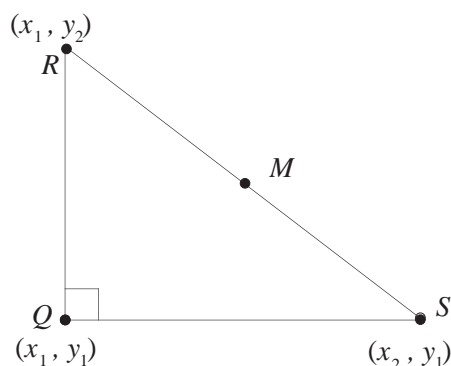


- 28. a.** Imagine a right triangle with its right angle sitting on the x -axis (not at the origin). Two of its vertices are $(0, 0)$ and $(2, 0)$. Find the coordinates of the third vertex when the hypotenuse is
- 5 units long;
 - 6 units long.
- b.** A different triangle has a hypotenuse 5 units long, one vertex at $(0, 0)$, and legs of equal length. Again the right angle vertex is on the x -axis but not at the origin. Find the coordinates of the second and third vertices of the right triangle.
- 29.** The vertices of $\triangle ABC$ are $A = (2, 1)$, $B = (4, 8)$, and $C = (6, -2)$. Find the length of the median from A to \overline{BC} .

- 30.** The vertices of $\triangle DEF$ are $D = (11, -1)$, $E = (13, 10)$, and $F = (3, 5)$.
- Show that $\triangle DEF$ is isosceles.
 - Find the length of the altitude from E to \overline{FD} .
- 31.** In the triangle from Problem 30, let $L = (7, 2)$. From Problem 30, we know that \overline{EL} is both a median and an altitude.
- Show that M , the midpoint of \overline{DE} , is equidistant from D , L , and E .
 - Show that $\triangle ELD$ is a right triangle.
- 32.** Pick four points that form a quadrilateral in a Cartesian plane. Find the midpoints of all four sides, and show that if you connect the midpoints, you get a parallelogram.
- 33.** Let the coordinates of the vertices of right triangle $\triangle QRS$ be (x_1, y_1) , (x_1, y_2) , and (x_2, y_1) . Show that M , the midpoint of the hypotenuse, is equidistant from the three vertices.

One way to show that a quadrilateral is a parallelogram is to show that the opposite sides are parallel, but there is *another* way that will be more useful here. What is it?

You've done this already for a specific right triangle in Problem 31.



TAKE IT FURTHER.....

If you call the length of the third side of the triangle d_1 , then you just have to show that d_2 , the length of the segment that connects the midpoints of the other two sides, is $\frac{1}{2}d_1$.

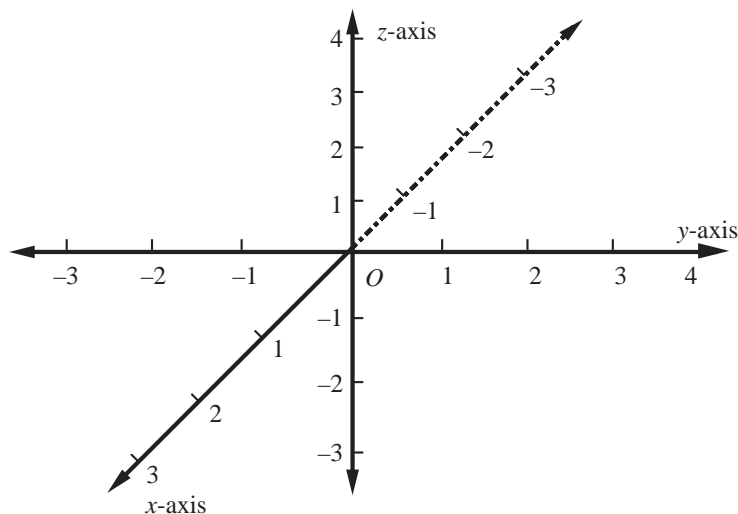
- 34.** Show that the length of the line segment whose endpoints are the midpoints of two sides of a triangle is one half the length of the third side of that triangle. Do this in a general way so that your proof will be valid for *all* triangles (use subscript notation).
- 35.** Use subscript notation to prove a general version of Problem 32.

COORDINATES IN THREE DIMENSIONS

“Conventional” here means that most people do it or write it this way.

The Cartesian coordinate plane is a system that aids mathematicians in talking about the relative positions of points, lines, and other two-dimensional objects or drawings. Points on three-dimensional objects may be represented by an ordered triple, (x, y, z) .

In mathematics, the conventional way to extend Cartesian coordinates to three dimensions is to add a third coordinate axis, perpendicular to the other two, and drawn like this:

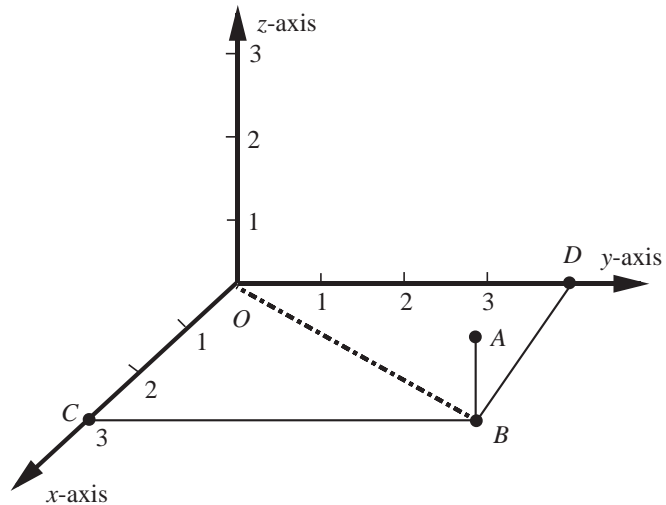


Imagine the x -axis “coming out of the page” with the positive half coming towards you and the negative half going away behind the page. The third axis is called the z -axis.

POINTS IN THREE-DIMENSIONAL SPACE

Remember that it is possible to devise systems of coordinates that are not the same dimension as the space they describe. But in the coordinate system used by Descartes, the dimensions *are* the same. Using the Cartesian system, the coordinate (3) names a unique point on the one-dimensional number line. The coordinates $(3, 4)$ name

a unique point in the two-dimensional plane. In three dimensions, it takes three coordinates to describe one particular point. In the picture below, B has coordinates $(3, 4, 0)$, and A has coordinates $(3, 4, 1)$.



You may want to experiment with a model of a 3D coordinate system if that helps you answer these questions.

No matter what point you look at in Problem 5, the x -, y -, and z -coordinates will always have the same value.

1.
 - a. Name three other points whose x -coordinate is 3 and y -coordinate is 4.
 - b. How many such points are there? Picture *all* of the points whose first two coordinates are $(3, 4)$. How are those points arranged? (That is, what shape is that collection of points?)
2. Picture all of the points whose z -coordinate is 3. Where are these points found? (The other two coordinates are unspecified, so they can take on all possible values.)
3. Name the shape made by all of the points whose first coordinate is the same as the second coordinate. (The third coordinate is left unspecified.)
4. Describe the set of points in three dimensions whose third coordinate is twice the second coordinate.
5. Describe all of the points whose coordinates satisfy the equation $x = y = z$. What shape do they make? Explain how you can get a solution to this problem from the answer to Problem 3.

6. **a.** Describe the set of points in three dimensions whose coordinates satisfy the equation $x = a$ (a constant). What is the shape of the graph of these points?
- b.** Describe the set of points in three dimensions whose coordinates satisfy the equations $x = -2$ and $y = 1$. What is the shape of the graph of these points?
7. Suppose you are given one fewer coordinate than you need! Here are three examples:
- a.** The points lie somewhere on a number line, but you are given no coordinates at all.
- b.** The points lie on a coordinate plane, but you only know that one coordinate is 3.
- c.** The points lie in 3-dimensional space, but you only know that the first two coordinates are 3 and 4.

How would you describe the *shape* of the collection of points that is described in each example?

8. Now suppose you are given two coordinates fewer than you need! Here are two examples:
- a.** The points lie somewhere in a coordinate plane, but you are given no coordinates at all.
- b.** The points lie in a coordinatized 3-dimensional space, but you are given only one coordinate, for example, 6.

How would you describe the *shape* of the collection of points that is described in each example?

If you systematize *how* you visualize something, that can be an important habit of mind.

The *xy*-plane is the plane containing the *x*- and *y*-axes. In this plane, the *z*-coordinate of every point is 0.

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9. **Write and Reflect** Describe *how* you visualized the figures in 6a and 6b. For part a did you use a 3D model, pick a particular value for a , and then imagine all possible y - and z -values to go with it? Or did you imagine the shape of $x = a$ in *two* dimensions and then extend it into three by allowing the z coordinate to take on all possible values? Did you do something else?
10. **Write and Reflect** Describe how you visualized the figure in Problem 4.
11. Take some shapes in the *xy*-plane: a point, a line, a circle, and a square. Now add the third dimension to each by allowing z to take on all possible values. What shapes result?

12. a. Name eight points that are 5 units from the origin in three dimensions.
- b. Picture all the points that are 5 units from the origin (in three dimensions). What shape do they make?
- c. Picture all the points that are 5 units from the x -axis. How are those points arranged? (That is, what shape do they make?)
- d. Picture all the points that are x units from the x -axis. How are those points arranged?

MORE CLASSROOM COORDINATES

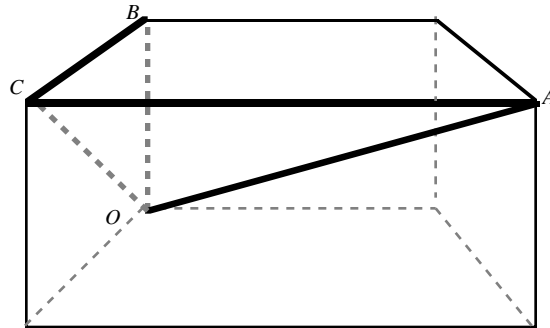
These questions ask you to design a 3D coordinate system for your classroom. Designate a corner of your classroom for the origin, and make sure the x -, y -, and z -axes run along the edges of the walls and floor so that the angle between any two axes is a right angle. Also, the axes should be positioned in the same way as the axes in the figure on page 40.

You may need to devise an appropriate measuring stick or have someone mark units with tape along the walls and floor for these problems.

13. Use your 3D coordinate system to describe the location of the four corners of one window.
14. a. Using the coordinate system you've set up in your classroom, locate any three objects (the light on the ceiling, the blackboard and so on); write down their coordinates.
- b. Exchange just the coordinates with another student. Try to find the three objects named by the list of coordinates you are given.

Perhaps your teacher will give you the necessary lengths and height of the walls that you will need for this problem.

15. Imagine a diagonal line running from the “origin” corner to the opposite corner of the ceiling, as in the picture below. To solve parts a and b below, you will need to know the height of the ceiling (OB) and the lengths of the two adjacent walls.



- What is the length of \overline{OC} ?
- What is the length of \overline{OA} ?
- What angle does \overline{OB} make with \overline{BC} ?

MIDPOINTS AND DISTANCE IN 3D

16. Point P has coordinates $(4, 5, 3)$. Figure out how far each of the following points is from P . Keep track of the thinking you use, and be sure you can explain your method in each case.
- $A = (4, 0, 3)$
 - $B = (4, 5, 0)$
 - $C = (0, 5, 0)$
 - $D = (0, 0, 3)$
 - $E = (-3, 6, -3)$
 - $O = (0, 0, 0)$

Explain in words or with a mathematical formula. You may explain using specific points like $P = (4, 5, 3)$ and $E = (-3, 6, -3)$ as an example, or you may be general, using points like $A_1 = (x_1, y_1, z_1)$ and $A_2 = (x_2, y_2, z_2)$.

17. Write a general rule for finding the distance between two points from their 3D coordinates.
18. Find the coordinates of the midpoints of the segments with the following endpoints:
 - a. $P = (4, 5, 3)$ and $B = (4, 5, 0)$
 - b. $P = (4, 5, 3)$ and $C = (0, 5, 0)$
 - c. $O = (0, 0, 0)$ and $P = (4, 5, 3)$
 - d. $P = (4, 5, 3)$ and $E = (-3, 6, -3)$
19. Write a general rule for finding the point midway between two points from their 3D coordinates.
20. A square with a one-unit sidelength sits with one vertex at the origin and two of its sides lying along the axes of a two-dimensional coordinate system. Its vertices have no negative coordinates. What are the coordinates of the farthest vertex from the origin? How far is that vertex from the origin?
21. A cube with a one-unit sidelength sits with one vertex at the origin and three of its edges lying along the axes of a three-dimensional coordinate system. Its vertices have no negative coordinates. What are the coordinates of the farthest vertex from the origin? How far is that vertex from the origin?
22. **Challenge** Why not extend the idea to four dimensions! A four-dimensional “hypercube” with a one-unit sidelength sits with one vertex at the origin and edges against the axes of a four-dimensional coordinate system. Its vertices have no negative coordinates. What are the coordinates of the farthest vertex from the origin? How far is that vertex from the origin?

SHAPES IN THE PLANE AND IN SPACE

This investigation is an opportunity to get more familiar with some basic shapes such as lines, circles, ellipses and triangles. This is a chance for students to gain a sense of some of the properties of the coordinates of these figures before moving on to operate on those coordinates.

CURVES IN TWO OR THREE DIMENSIONS

1. Here's a recipe to perform on a point: square each coordinate and add the results. Find eight points that produce a result of 5 when you do this to them. What figure do you get if you look at all the points that produce 5? Find eight points that produce each of the following results:
 - a. 13
 - b. 625
 - c. 100
 - d. 169
 - e. 1

..... **WAYS TO THINK ABOUT IT**

One way to go about this problem is to use a calculator. If you want a point for which the recipe produces 5, you could just *decide* that one of the coordinates of the point is 1.5, square that number, subtract the result from 5 to see what's left, and take the square root of that.

Another way is to think about sums of square integers. Can *every* integer be expressed as the sum of two squares? Can 5 be expressed as the sum of two squares? In Problem 1, we've made life easy by picking numbers that *can* be expressed as the sum of two squares.

.....

2. a. If you drew the picture of all the points in the plane that are 13 units from the point $(4, -1)$, what figure would you get?

- b.** Find the coordinates of eight points that are 13 units from the point $(4, -1)$. How can you tell if some new point is on this figure by doing a little calculation on its coordinates? (How can you tell if the point (a, b) is on the figure?)
- 3.** Suppose you perform this recipe on points in three dimensions: Take each coordinate, square it, and add the results.
- a.** Find eight points that produce 81 when you perform the recipe. What do you get if you look at *all* points that produce 81?
- b.** What do you get if you look at all points in three dimensions that are 7 units from the origin? Find the coordinates of two or three points that are each 7 units from the origin.
- 4.** Let's go back to 2 dimensions. Here's another recipe to perform on the coordinates of points:
- square the x -coordinate;
 - square the y -coordinate;
 - multiply the x -coordinate by the y -coordinate;
 - add the results of the first three steps above.

What values do you get if you perform this recipe on each of the following points?

- a.** $(10, 0)$
- b.** $(0, -10)$
- c.** $(6, 8)$
- d.** $(6, -8)$
- e.** $(-5, 5\sqrt{3})$
- f.** $(5, 5\sqrt{3})$

Make sure to list at least six points, not less. Try points with coordinates of value 0, of course, but try something like $(1, -1)$ too.

- 5.** Find six points that produce 1 when you apply the recipe in Problem 4. What do you get if you look at *all* the points that produce 1?

6. Try this recipe:

- square the x -coordinate and multiply by three;
- square the y -coordinate;
- add the results.

Find six points that produce 4 when you apply this recipe. What do you get if you look at *all* the points that produce 4?

7. Here are four related recipes with numerical outcomes. What shapes do they produce?

- a. The sum of the coordinates is 12.
- b. The difference of the coordinates is 12.
- c. The product of the coordinates is 12.
- d. The ratio of the coordinates is 12.

8. Make up your own recipe. Keep it fairly simple.

- a. See what happens when you apply that recipe to the six points listed in Problem 4.
- b. Pick what seems like a suitable goal number, and find six points that produce that number when you apply your recipe. Sketch the shape that describes *all* points that produce that number.
- c. **Challenge** Trade sketches with someone else. See if each of you can figure out the other person's recipe, just by looking at the shape it produces.

TRIANGLES IN THE PLANE

- 9. If two vertices of an equilateral triangle lie in the Cartesian plane at $(1, 0)$ and $(9, 0)$, find coordinates for the third vertex.
- 10. Consider the six points $A = (5, 1)$, $B = (10, -2)$, $C = (8, 3)$, $A' = (2, 3)$, $B' = (7, 0)$, and $C' = (5, 5)$. Show that $\triangle ABC \cong \triangle A'B'C'$.
- 11. Consider the six points $A = (-800, -500)$, $B = (160, 12)$, $C = (-737, -484)$, $A' = (0, 0)$, $B' = (3840, 2048)$, and $C' = (252, 64)$. Show that $\triangle ABC \sim \triangle A'B'C'$.

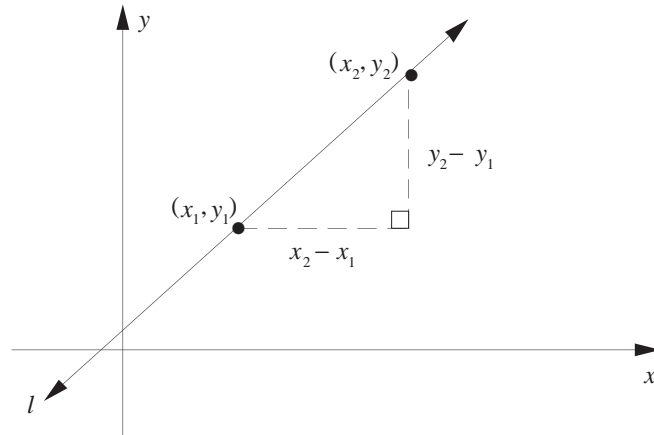
- 12. Write and Reflect** Explain how to tell if two triangles are congruent by calculations on the coordinates of their vertices.
- 13. Write and Reflect** Explain how to tell if two triangles are similar by looking at the coordinates of their vertices.

LINES

- 14.** Do you remember Problem 7 from Investigation 5.3, where you drew the picture of all the points for which the x -coordinate was the same as the y -coordinate? Now plot several points whose y -coordinates are 3 more than their x -coordinates. Is there any regularity to the points? Explain.
- 15.** Plot several points whose y -coordinates are 2 more than their x -coordinates. Draw the picture of *all* the points with this property.
- 16.** Plot several points whose y -coordinates are 1 more than their x -coordinates. Draw the picture of *all* the points with this property.
- 17.** Plot several points whose y -coordinates are
- a.** twice their x -coordinates;
 - b.** three times their x -coordinates;
 - c.** four times their x -coordinates.

SLOPE

An important property of a line is its *slope*. The slope of a line is a numerical measurement of its steepness. Let l be a line through two points (x_1, y_1) and (x_2, y_2) .



We define the slope of l as follows:

$$\text{slope of } l = \frac{y_2 - y_1}{x_2 - x_1}.$$

Look for places where you can use slope in solving Problems 18–24 and also problems that appear later in the module.

- 18. Write and Reflect** Some of the lines you sketched had points whose coordinates were of the form $(x, x + \text{something})$. What did those lines have in common?
- 19. Write and Reflect** Some other lines you sketched had points whose coordinates were $(x, x \times \text{something})$. What did *those* lines have in common?

COLLINEARITY

- 20.** Find a point that is collinear with $A = (5, 1)$ and $B = (8, -3)$. Explain how you did it.
- 21.** Give a set of instructions for finding points that are collinear with $A = (5, 1)$ and $B = (8, -3)$. Try your instructions out on a friend. Explain why your method works.
- 22.** Is $P = (-4, -14)$ collinear with $R = (-40, -30)$ and $S = (80, 20)$? Explain.
- 23. Write and Reflect** Give a set of instructions for testing points that are collinear with $R = (-40, -30)$ and $S = (80, 20)$. Pick some points that are collinear

with R and S and some that are not. Ask a friend to test the points for collinearity using your method. Explain why your method works.

This is a very important problem!

- 24. Write and Reflect** Suppose R and S are two points. Give a set of instructions for testing a third point P to see if it is collinear with R and S . Explain why your method works.

TAKE IT FURTHER.....

- 25.** Let \mathcal{C} be the circle whose center is at the origin of the plane and whose radius is 5.
- a.** Find any points on \mathcal{C} that are also on the vertical line that contains $(4, -9)$.
 - b.** Find the intersection of \mathcal{C} with the vertical line that contains $(3, 2)$.
 - c.** Find any points on \mathcal{C} that are also on the horizontal line that contains $(8, 0)$.
 - d.** Find the intersection of \mathcal{C} with the horizontal line that contains $(8, 3)$.
 - e.** Find any points on \mathcal{C} that are 13 units from the origin.
 - f. Challenge** Find any points on \mathcal{C} that are 13 units from $(8, 16)$.
- 26.** If you had to build a regular tetrahedron using the equilateral triangle described in Problem 9 as the base, what would be the coordinates of the fourth vertex?

INTRODUCTION TO COORDINATES AND VECTORS

Read about Descartes' life and contributions in the "Perspective on Descartes" essay in Investigation 5.2.

This quotation can be found in *Mathematics in Western Culture* by Morris Kline, Oxford University Press, 1953, p. 159.

Descartes' contribution of combining algebra and geometry was a major breakthrough in mathematics. As the 18th-century French mathematician Joseph-Louis Lagrange said:

"As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company, they drew from each other fresh vitality and thenceforward marched on at a rapid pace toward perfection."

Of course, even a mathematician as great as Lagrange could not anticipate all of what was yet to come. If he had, he might have chosen some term other than "perfection," because by the middle of the 19th century, there were even more methods that integrated algebra and geometry. One of these new inventions was the language of vectors.

In this section of the module, you will study the mathematics behind coordinates and vectors. Coordinates provide a language that emphasizes *location* in space, allowing you to calculate distances and directions and to verify all kinds of attributes of figures based on the placement of points. Vectors emphasize distance and direction, without much (or any) attention to location. Both languages provide powerful tools for deciding when things are equivalent and when they are not.

"Do something" might mean to perform an arithmetic operation. For example, you might add some number to the coordinates of a figure or multiply them by some number. Or you might shrink or stretch a figure by some factor. That's "doing something."

Investigations 5.8–5.12 explore a two-part question:

- When you "do something" to the coordinates of a figure, what happens to the figure?
- When you do something to a figure, what happens to its coordinates?

CHECKPOINT.....

If you haven't just finished the first section of this module (Investigations 5.1–5.6), here are some questions to get your coordinate sense back in shape. Make sure you draw pictures as you answer them.

1. Plot the points whose coordinates are $(3, 4)$, $(4, 4)$, $(8, 4)$, $(-2, 4)$, and $(0, 4)$. Plot three other points whose y -coordinate is 4. Plot all points whose y -coordinate is 4.
2. Sketch all the points whose y -coordinate is -2 . Sketch all the points whose x -coordinate is 5.

Once we choose a coordinate system with two perpendicular axes, we declare one axis to be the “horizontal” direction and call the other “vertical.” A *vertical line* is then any line parallel to the vertical axis. By convention, in a typical Cartesian coordinate system, the y -axis is taken to be vertical.

3. Draw a vertical line through the point with coordinates $(-2, 1)$. Identify six other points on your line. What can you say about every point on your line?
4. Name and plot seven points whose first coordinate is the same as the second coordinate. Graph all the points whose first coordinate is the same as the second coordinate.
5. Graph all the points whose first coordinate is the negative of the second coordinate.
6. Find and identify eight points that are 5 units away from the origin. Sketch the picture of all the points that are 5 units from the origin.
7. Suppose $P = (12, -5)$. How far is P from the origin? How far is P from $(15, -9)$?
8. Describe in simple language a method for finding the distance between two points if you know their coordinates.
9.
 - a. Suppose $A = (8, -12)$ and O is the origin. What are the coordinates of the midpoint of \overline{AO} ?
 - b. Suppose $C = (3, 5)$ and $D = (-9, 10)$. If D is the midpoint of \overline{CE} , find the coordinates of E .
10. Describe in simple language:
 - a. A method for finding the coordinates of a segment’s midpoint if you know the coordinates of its endpoints;
 - b. A method for finding an unknown endpoint if you know the coordinates of a segment’s midpoint and one endpoint.

STRETCHING AND SHRINKING THINGS

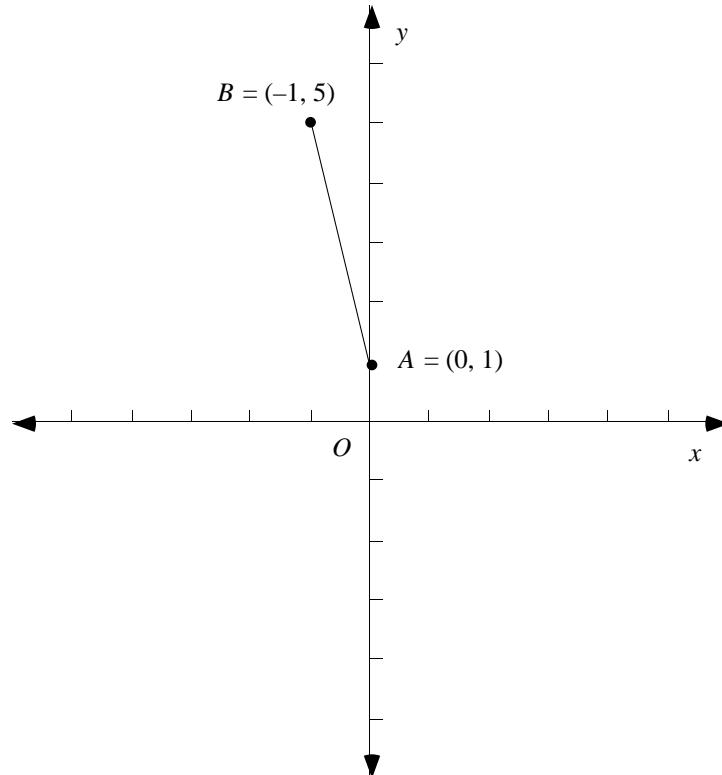
Here are some experiments to try that may give you ideas about different ways coordinates can help you investigate shapes.

Answer this question if you used the Connected Geometry module A Matter of Scale: What does this have to do with dilations?

You now have several pentagons. Are your corresponding vertices collinear?

1. On a Cartesian coordinate system, draw a pentagon whose vertices are (8, 5), (10, 3), (12, 6), (12, 9), and (9, 9). Connect each vertex to the origin, and find the midpoints of all these lines. Connect these midpoints to make a smaller pentagon.
 - a. How would you describe the relationship between the two pentagons?
 - b. How would you describe the coordinates of the small pentagon's vertices in terms of the coordinates of the big pentagon's vertices?
 - c. Use the coordinates of your original pentagon's vertices to find the vertices of a pentagon whose sidelengths are twice as long as the original's. Draw this new pentagon.
 - d. Use the coordinates of the vertices of your original pentagon to find the vertices of a pentagon whose sidelengths are one third as long (the numbers won't always be neat, but that's life). Draw this new small pentagon.
2.
 - a. Plot the four points (0, 2), (2, 3), (3, 5), and (5, 1) on a coordinate system and connect them to make a quadrilateral. For each point, multiply both the first and second coordinates by 4; plot the result. Connect these new points to get a quadrilateral. How is it related to your original?
 - b. Multiply the coordinates of your original quadrilateral by -2 or -1 and plot what you get. How is this figure related to your original?
 - c. Multiply the first coordinates by 2 and the second coordinates by 3. How is this figure different from the figures in parts a and b?
 - d. **Challenge** Start with the four points given in part a. When would a new quadrilateral be entirely inside the original, and when would it contain the original? When would a new quadrilateral, created by multiplying both coordinates of each point by the same number, overlap with the first quadrilateral, and when would the two be disjoint?

3. In the picture below, \overline{AB} is one side of a square.



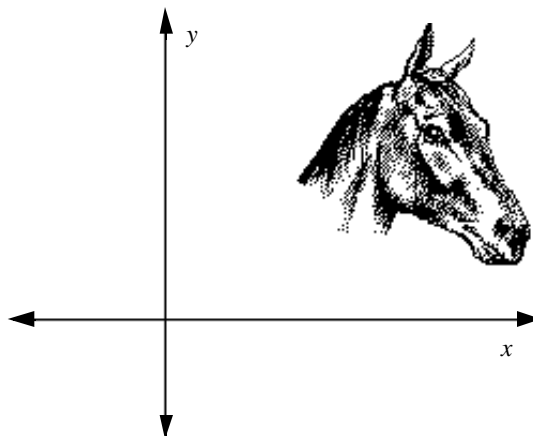
In Problem 2, where the sides of the resulting quadrilateral are three times as long as the sides of the original, we'd say that the second quadrilateral is *scaled* by three. If you put both quadrilaterals on the wall, the larger one would take three times as much height and length on the wall as the original.

You might want to break this down into several simpler rules: What happens when the number is $\geq 1, \dots$?

- Find two other points C and D so that $ABCD$ is a square.
 - Multiply both coordinates of each vertex of your square by $\frac{1}{2}$. Plot these new points and connect them. Describe what you get and how it's related to the original square.
 - Multiply both coordinates of each vertex of your original square by -3 . Plot these new points and connect them. Describe what you get and how it is related to the original.
4. Write a rule that describes what happens to a figure when you multiply both coordinates of each point by some number.

You might want to scale just some important features like the nose, ears, and mouth.

5. Here's a picture on a coordinate system of the head of Trig.
 - a. Use coordinates to scale Trig by 2.
 - b. Use coordinates to get a scaled copy (or outline) of Trig that's upside down.



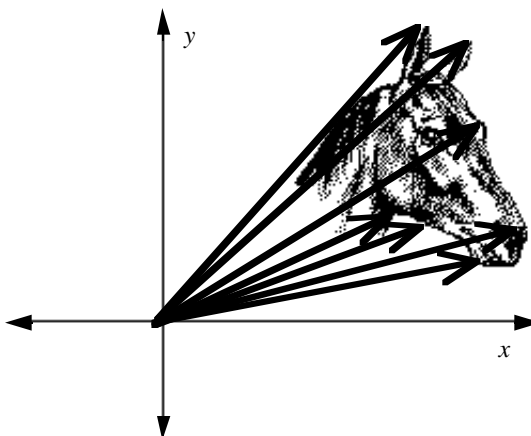
..... WAYS TO THINK ABOUT IT

The coordinate method for stretching and shrinking Multiplying both coordinates of a figure's vertices by some positive number n , greater than or equal to 1 (and then reconnecting those vertices) reproduces the figure or produces a "blown-up" copy (an enlargement, similar to the original), located n times as far from the origin. Multiplying the coordinates by n where $0 \leq n < 1$ produces a reduction in the size of the figure, and when $n < 0$, the figure will be enlarged or reduced, but reflected (flipped) over the origin.

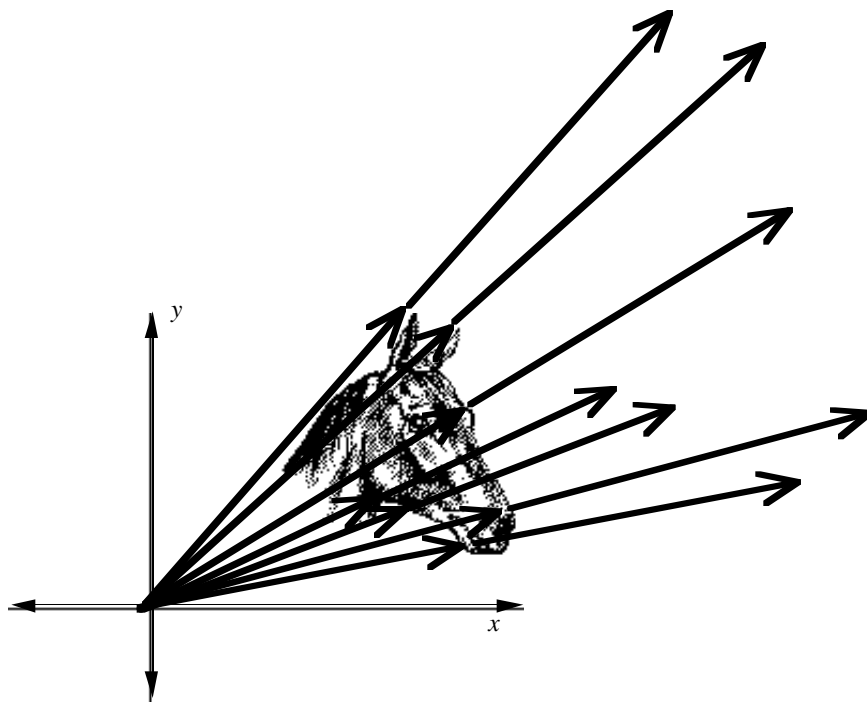
You may have met vectors before. If not, just think of a vector as an "arrow." It's a line segment with a particular direction and length. You'll find out more about vectors later in this module.

The vector method for stretching and shrinking A student in one of the field test sites for *Connected Geometry* came up with what she called the "vector method" for making a blow up of Trig's picture. Here's what she did:

On a copy of the picture of Trig, she drew arrows (“vectors”) from the origin to interesting places on Trig’s head (tips of the ears, end of the nose, and so on):

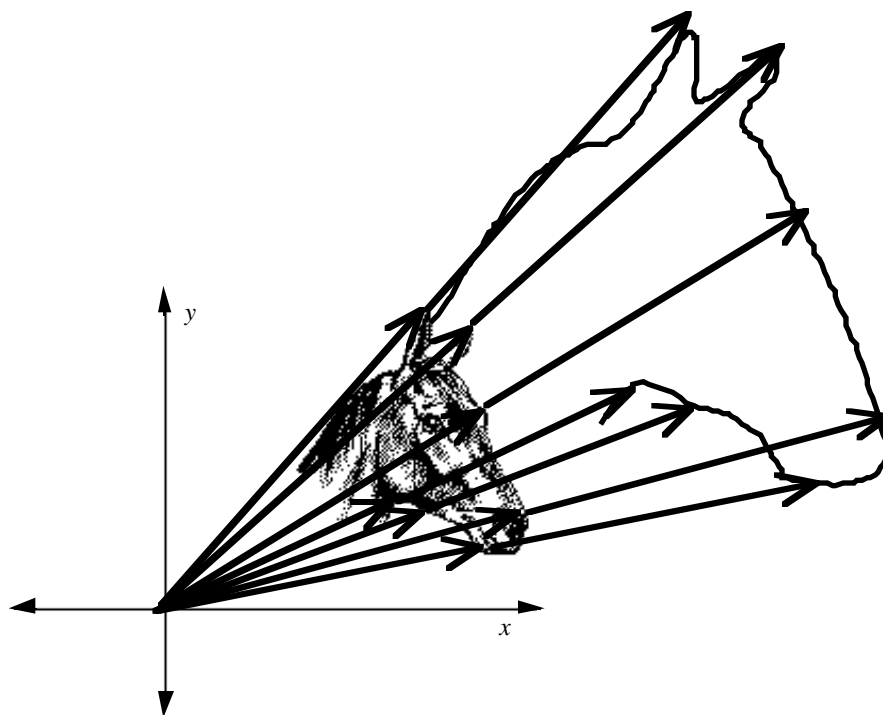


Then she “stretched” each vector, keeping its exact same direction and doubling its length. She did this by placing a ruler alongside each vector to keep the same direction, and then multiplying the vector’s length by 2.

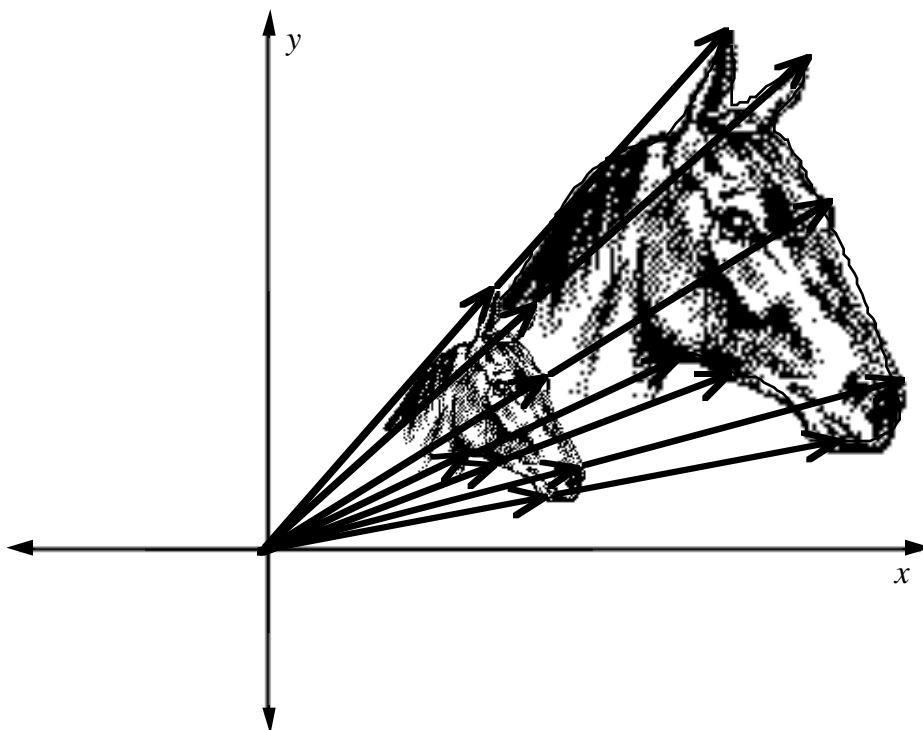


This gave an outline for “Big Trig.” Then by adding more vectors, she was able to get a pretty good sketch:

She used many more vectors than this—maybe two or three times as many.



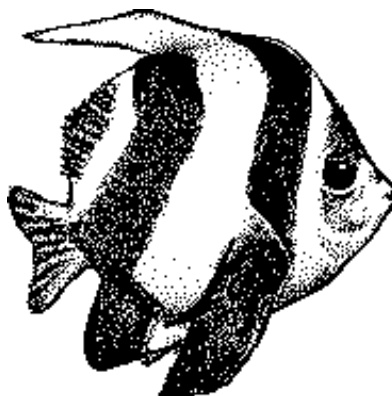
Then she added the details.
This student was a very
good artist.



.....

6. Try the vector method on another picture, but this time scale it down by a factor of 2, so the lengths in the new picture will only be $\frac{1}{2}$ as long as those in the original.

Choose your own picture or
use this fish.



7. Compare the vector method of Problem 6 to the coordinate method of Problem 5. How are the coordinates of the corresponding points in Problem 5 related to each other? How are the coordinates of the “arrowhead” of each original vector in Problem 6 related to the coordinates of the arrowhead of the corresponding shrunk vector?
8. How would you adapt the vector method to include negative numbers? Try it: scale one of the pictures or figures used in this investigation by -2 .

CHANGING THE LOCATION OF THINGS

You could trace the pentagon on another piece of paper and then slide it. Or you could measure with a ruler, or you could “do something” to the coordinates and graph a new pentagon.

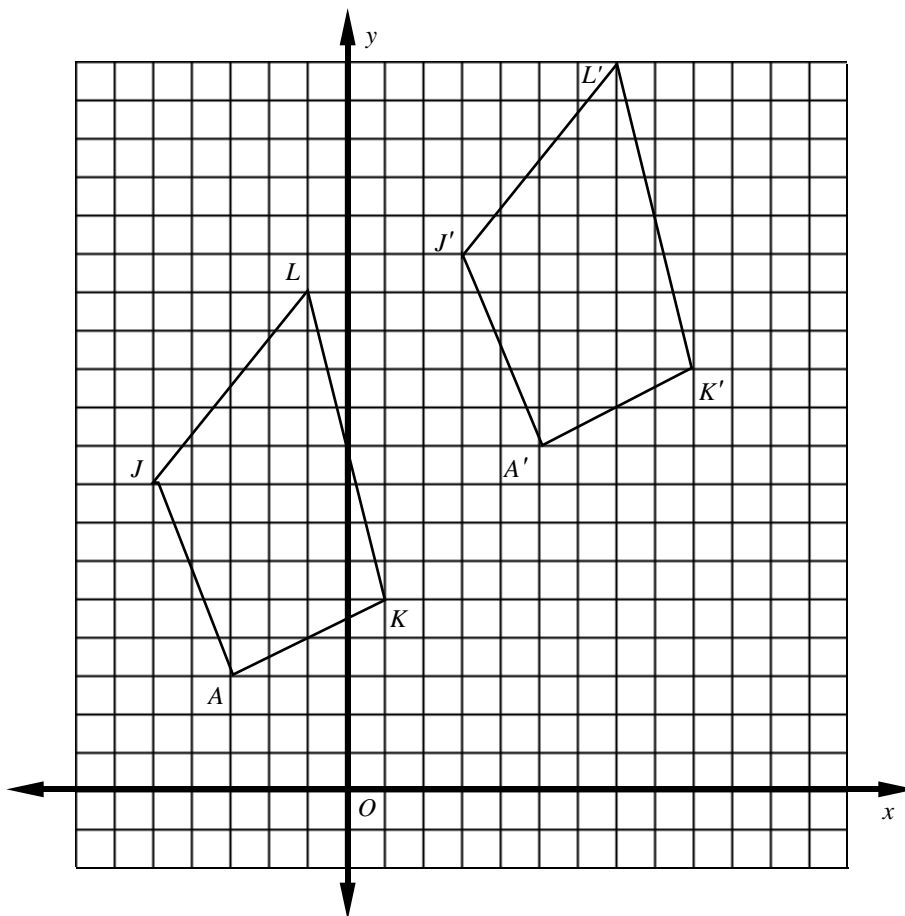
Try it. Slide the original up and then compare coordinates.

“Specifies” here means “The rule tells (or describes) how the figure changes location.”

1. On a coordinate system, draw a pentagon whose vertices are $(8, 5)$, $(10, 3)$, $(12, 6)$, $(12, 9)$, and $(9, 9)$. Do something to the coordinates to move the pentagon over 8 units to the right (not up or down, just over).
 - a. Describe the relationship between the two pentagons.
 - b. Describe the coordinates of the new pentagon’s vertices in terms of the coordinates of the vertices of the old pentagon.
2.
 - a. Now slide the original figure *up* 8 units. How would you describe the new set of coordinates?
 - b. Now slide the original right 6 units and up 8. Compare coordinates with the original.
3. Describe the effect on your original pentagon if you take the vertices and
 - a. add 10 to the first coordinates and add 7 to the second coordinates;
 - b. add 10 to the first coordinates and subtract 7 (or add -7) from the second coordinates.
4. Make up a rule that specifies what happens to a figure when you add some number to all the first coordinates and add some other number to all the second coordinates of the vertices.

Mathematicians say that $AKLJ$ has been translated to $A'K'L'J'$.

5. Refer to the picture below.



- Describe what you have to do to $AKLJ$ to get $A'K'L'J'$.
- Describe what you have to do to the *coordinates* of the vertices of $AKLJ$ to get the *coordinates* of the vertices of $A'K'L'J'$.

One easy way to say “take some points, add 10 to the first coordinates and add 6 to the second coordinates” is to say “send each point (x, y) to the point $(x + 10, y + 6)$ ” or even “ $(x, y) \mapsto (x + 10, y + 6)$.”

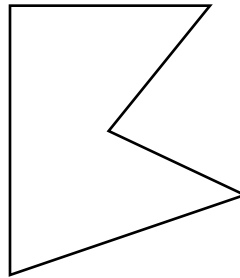
6. Apply the rule $(x, y) \mapsto (x + 10, y + 6)$ to the vertices of a triangle; then connect the three new points. What figure do you get?

When you see the symbol \mapsto , say “maps to.”

When drawing your own figures, it helps to use graph paper and make the vertices integers.

Don't forget the "Describe . . . " part of this problem.

7. How would you write the rule "take the points and multiply the coordinates by 2" using the $(x, y) \mapsto (\text{blah}, \text{blah})$ notation?
8. Draw a polygon on a coordinate system. If you like, you can make your polygon look roughly like this:



Apply each rule below to the vertices of the polygon and draw the resulting polygon. (You might try applying the rules to more than one polygon.) Describe how each resulting polygon is related to the original.

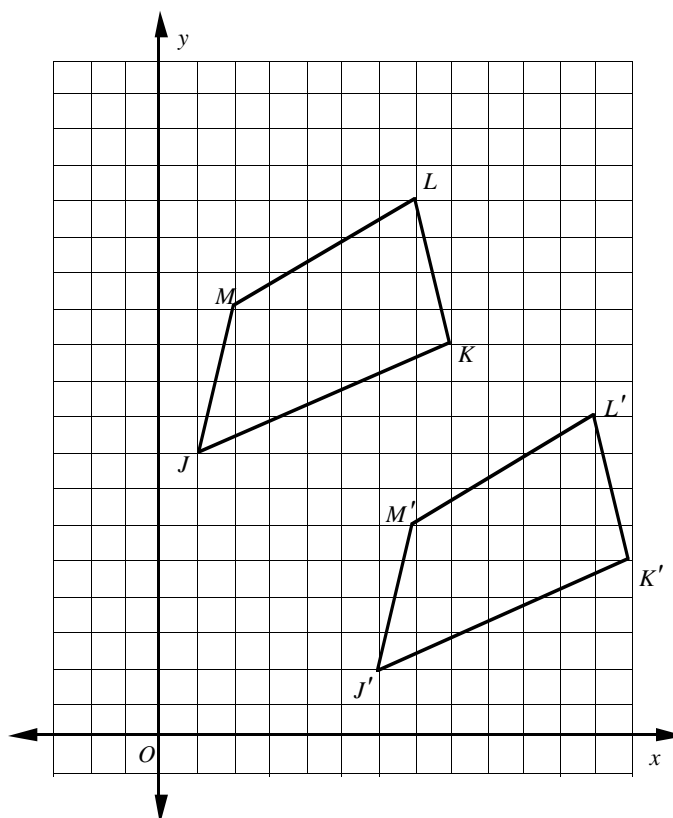
- a. $(x, y) \mapsto (x + 8, y + 5)$
- b. $(a, b) \mapsto (a - 8, b + 5)$
- c. $(a, b) \mapsto (3a, 3b)$
- d. $(x, y) \mapsto \left(\frac{x}{2}, \frac{y}{2}\right)$
- e. $(x, y) \mapsto \left(\frac{x}{2} + 7, \frac{y}{2} + 10\right)$
- f. $(x, y) \mapsto (-x, y + 2)$
- g. $(x, y) \mapsto (2x, y + 2)$

PICTURES FROM RULES, RULES FROM PICTURES

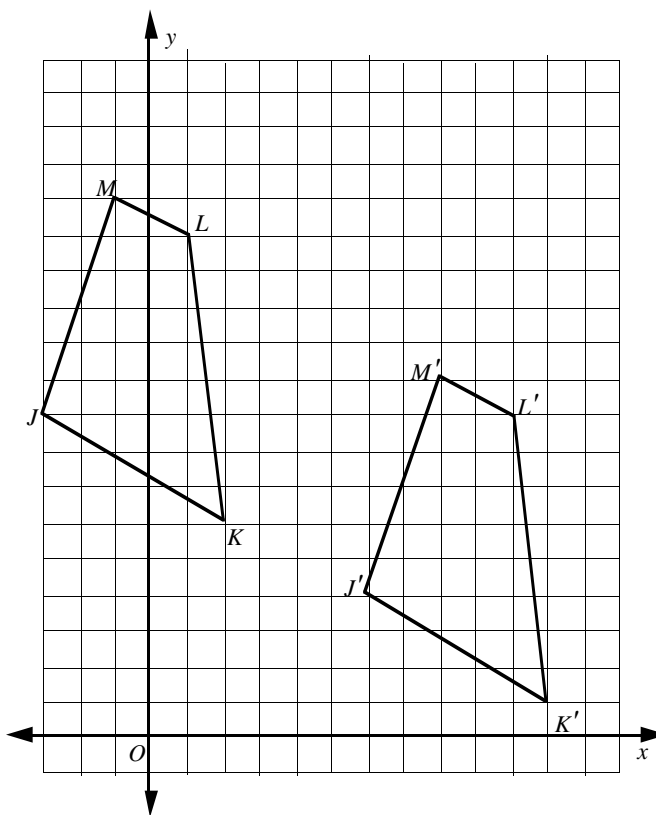
If you get stuck, label all the coordinates and look for a way to change the coordinates of one polygon into the coordinates of the other.

- For each of these pictures, write a rule that will translate the vertices of polygon $JKLM$ to polygon $J'K'L'M'$. (Use the \mapsto notation used in Problem 8 of Investigation 5.9.)

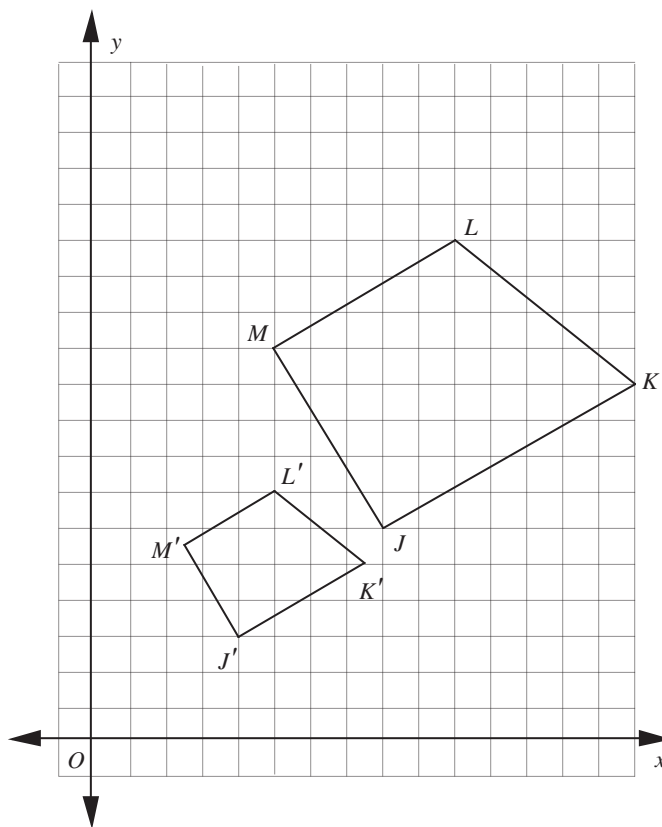
a.



b.

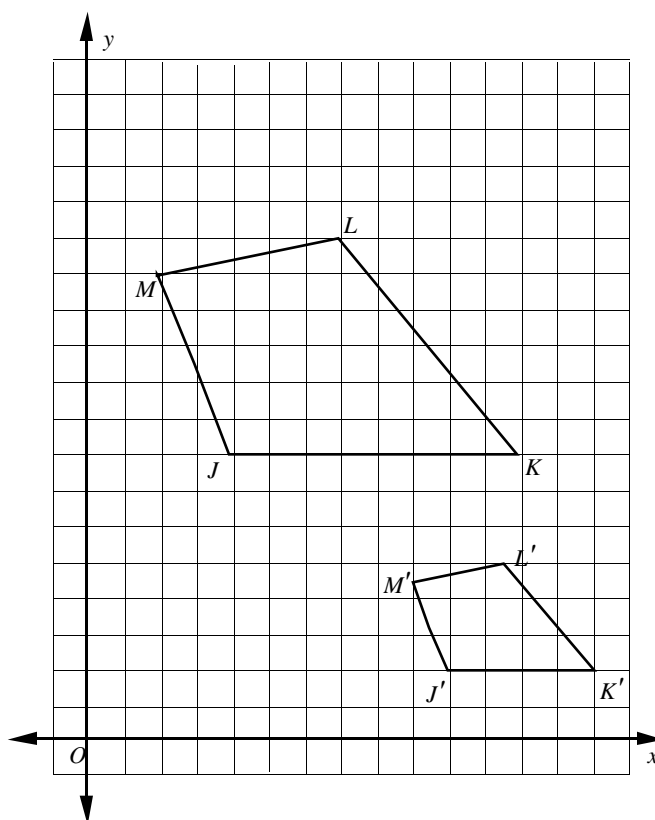


c.



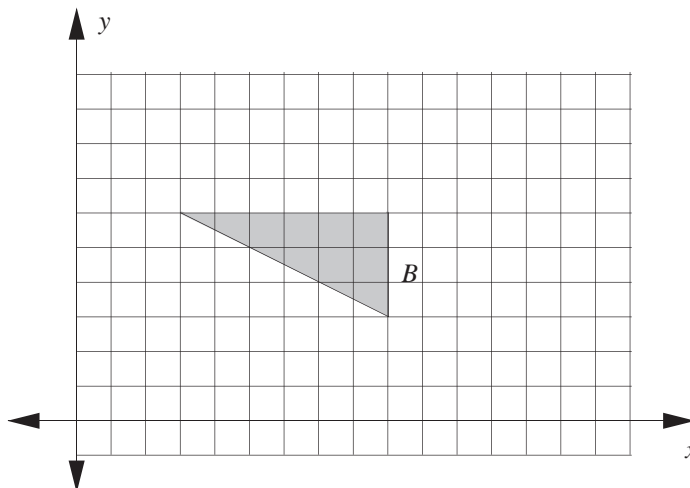
Hint: Two steps

d.



2. In $\triangle DIG$, $D = (4, 3)$, $I = (8, -2)$, and $G = (5, 7)$. In $\triangle RAT$, $R = (9, 2)$, $A = (13, -3)$, and $T = (10, 6)$. Describe the movement(s) that takes $\triangle DIG$ to $\triangle RAT$. Describe the arithmetic that changes the coordinates of the vertices from those of $\triangle DIG$ to those of $\triangle RAT$.

3. Both Jorge and Yutaka are asked to transform $\triangle B$ by scaling it by $\frac{1}{4}$ and translating it 16 units to the right, but they're not told in which order to perform the transformation. Jorge chooses to scale the triangle first and then translate it. Yutaka does the translating first and then the scaling.

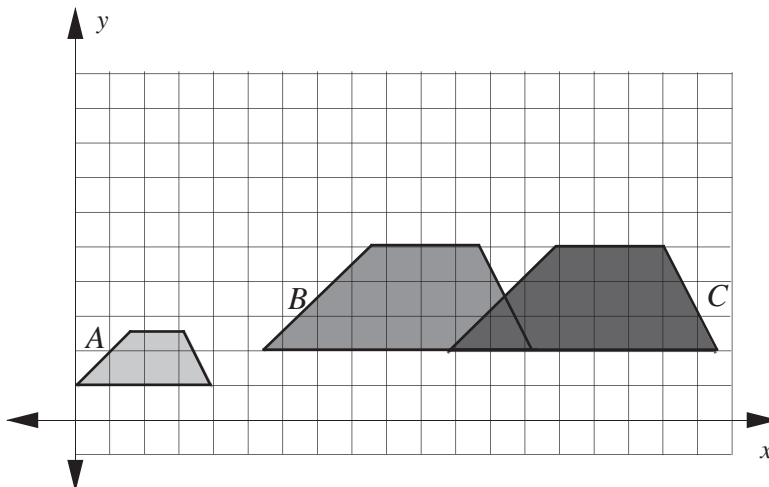


- Pick a vertex or two of triangle B . Follow Jorge's and Yutaka's methods and write down where the vertices end up.
- Do both methods end up with the same triangle in the same place?
- Below are algebraic statements that describe Jorge's and Yutaka's solutions. Which is Jorge's, and which is Yutaka's?

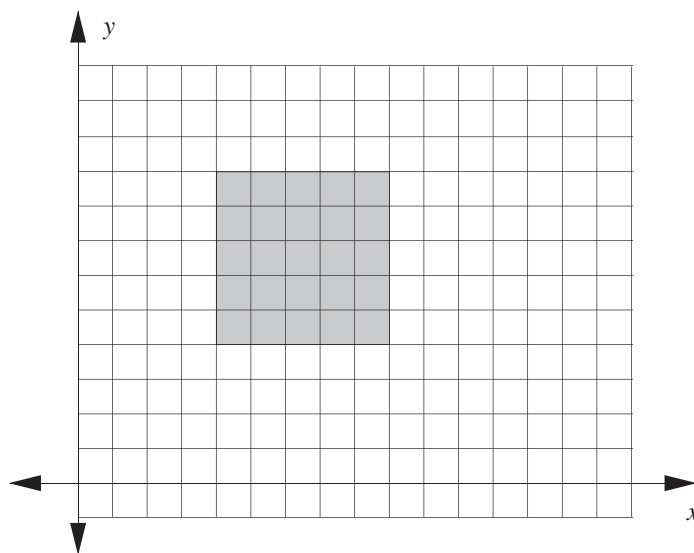
What could $\frac{1}{4}(x, y)$ mean?

- $(x, y) \mapsto \frac{1}{4}(x, y) + (16, 0)$
- $(x, y) \mapsto \frac{1}{4}(x + 16, y)$

4. Trapezoid A is scaled and translated using two different rules. Match each of the two new trapezoids to the rule that describes it.



- $(x, y) \mapsto 2(x, y) + (7, 0)$
 - $(x, y) \mapsto 2(x + 7, y)$
5. Copy the square onto graph paper; then apply each of the rules below to the vertices of the square and draw the resulting figures. Indicate the shape of the resulting figure—is it a square, a rectangle, or something else?



- a. $(x, y) \mapsto (x + 8, y + 5)$
- b. $(x, y) \mapsto (x - 3, y)$
- c. $(x, y) \mapsto (3x, 4y)$

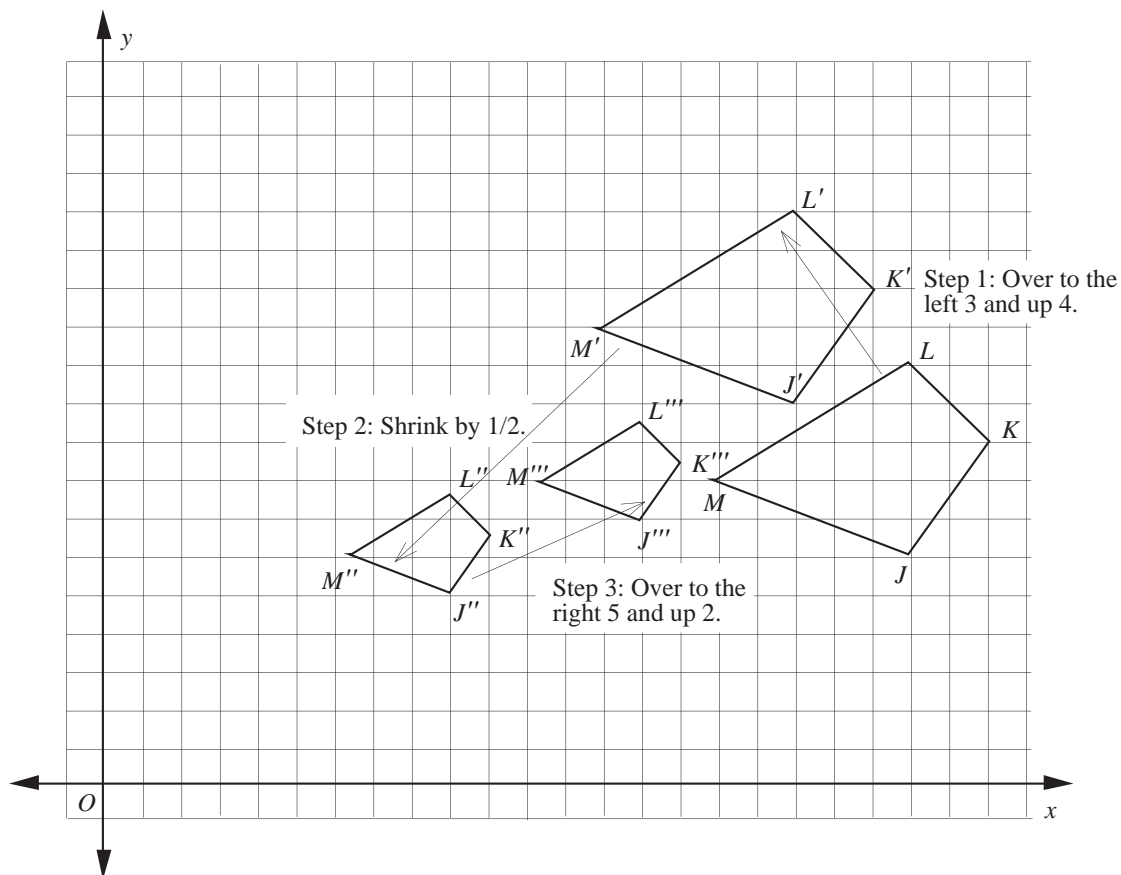
- d. $(x, y) \mapsto (-x, -y)$
- e. $(x, y) \mapsto (-2x, y - 2)$
- f. $(x, y) \mapsto (2x + 1, -2y)$
- g. $(x, y) \mapsto (x, 2(y + 1))$
- h. $(x, y) \mapsto (y, x)$
6. a. Classify the rules in Problem 5 as either rules that will take a square and result in a square, or that will take a square and result in a rectangle. What would be the result of $(x, y) \mapsto (\frac{1}{2}x, \frac{1}{2}(y + 2))$?
- b. **Challenge** Can a rule like those above—multiplying the coordinates by a fixed number and/or adding a fixed number to the coordinates—change a square into something that is *not* a rectangle?
7. Here's a more complicated rule:

$$(a, b) \mapsto \left(\frac{1}{2}(a - 3) + 5, \frac{1}{2}(b + 4) + 2 \right).$$

One student, Stella, responded to this problem with,

“Well, I think I see a different way to write it. First you move over to the left 3 and up 4. Then you shrink by a half. And then you move over to the right 5 and up 2. So I'd write it like this:

$$\frac{(a, b) + (-3, 4)}{2} + (5, 2).”$$



- a.** Is she right? Check it out with some coordinates. That is, find the coordinates of a point (let's use J), and do

$$(a, b) \mapsto \left(\frac{1}{2}(a - 3) + 5, \frac{1}{2}(b + 4) + 2 \right)$$

to it, plotting what you get. Then follow Stella's directions (the three steps). Do you end up in the same place?

- b.** Describe the rule in English in a way that involves only two steps. Then write your new rule using mathematical notation.

CHECKPOINT.....

Check your description with some pictures: Draw a picture on a coordinate system, apply the rule to some points on the picture, and see if your description is right.

8. Describe in words what this rule does:

$$(x, y) \mapsto \left(\frac{1}{2}(2(x - 4) + 6), \frac{1}{2}(2(y + 3) - 4) \right).$$

9. Here's a rule written in English:

To move any point P , draw a line segment from P to $(3, 4)$, and plot the midpoint.

Draw a few pictures to see what the rule does. Find a way to describe this in the $(a, b) \mapsto (\text{blah}, \text{blah})$ style.

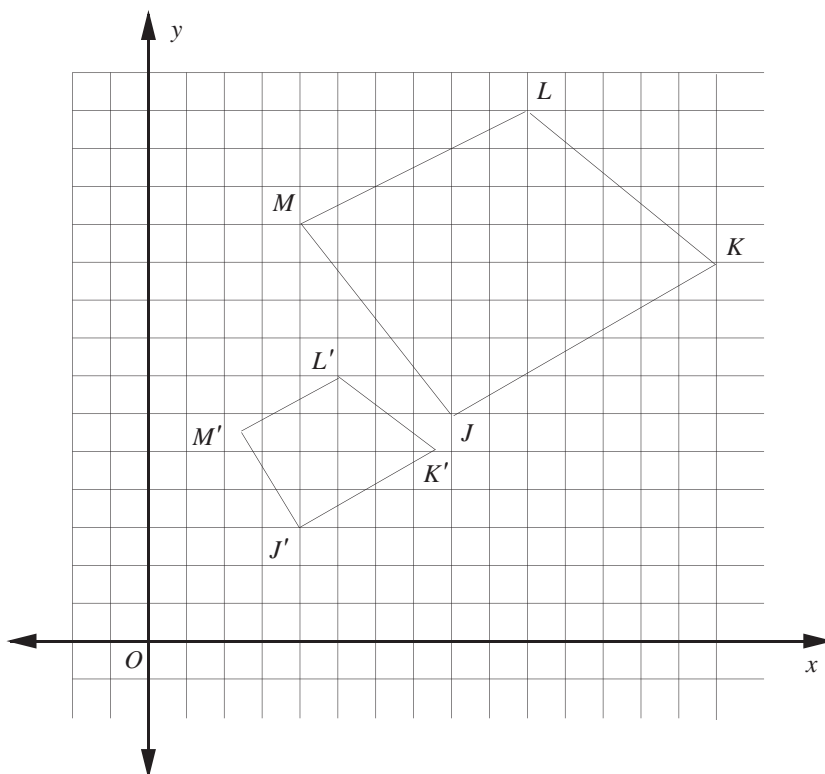
Use a corner of your room as the origin. Show what each rule does to a particular point.

10. So far, we have been working in two dimensions (the xy -plane). Describe in words what the following rules do to figures in space:
- $(x, y, z) \mapsto (3x, 3y, 3z)$
 - $(x, y, z) \mapsto (\frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z)$
 - $(x, y, z) \mapsto (3 + x, 2 + y, z - 1)$
 - $(x, y, z) \mapsto (\frac{1}{2}(2(x - 4) + 6), \frac{1}{2}(2(y + 3) - 4), \frac{1}{2}(2(z + 2) - 1))$
11. **Write and Reflect** Take the rule in Problem 8 and simplify it either in words, or in mathematical notation. Show an intermediate form of the rule (that is, show a step in between the rule as written in Problem 8 and its final form). Check your simplified rule by applying it to the same points you used in Problem 8. Also compare $(x, y) \mapsto (2x + 2, 3y + 3)$ and $(x, y) \mapsto (2(x + 1), 3(y + 1))$. Do these rules have the same result?

TAKE IT FURTHER......

- 12.** Explain why two rules can *do* different things but have the same end result. Give an example of two rules that do different things but end up with the same results.

In Investigation 5.10, you found a rule that would send $JKLM$ onto $J'K'L'M'$:



One way to do this is to multiply all the coordinates by $\frac{1}{2}$ so that your rule looks like $(x, y) \mapsto (\frac{x}{2}, \frac{y}{2})$. There's another way to write this, which was introduced in Investigation 5.10:

$$(x, y) \mapsto \frac{1}{2}(x, y).$$

Try it at lunch today or tomorrow.

This makes sense. If you walk up to someone and ask, “What’s 5 times the point $(2, -3)$?” you will probably get a response of $(10, -15)$, even from someone who makes up the answer on the spot. It turns out that $5 \cdot (2, -3) = (10, -15)$ is a very useful way of scaling points in geometry.

'Scaling a point' is shorthand for "dilating a point with center at the origin." To make sense, we need to have chosen an origin and a scale factor.

Also, just as in algebra, most people leave out the sign with this kind of notation, and write $6(4, 1)$ instead of $6 \cdot (4, 1)$.

This problem asks you to do seven calculations. There's a reason for this.

A scalar multiple of B is a point that you get by scaling B by some number. Of course you can't calculate every possible scalar multiple of B , but you can probably figure out what the set of all scalar multiples of B looks like after you calculate a few.

DEFINITION

To **scale a point by a number**, multiply each of the point's coordinates by the number. For example:

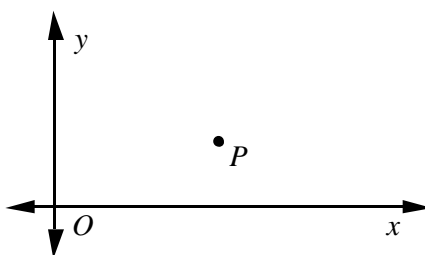
$$6(4, 1) = (24, 6) \text{ and } -2(3, 1, -4) = (-6, -2, 8).$$

1. For each problem, locate A and cA on a coordinate system (cA is just the point you get when you scale A by c). Use the same coordinate system for all the problems. Describe any patterns you see in your picture. How is the location of cA related to the location of A ?
 - a. $A = (5, 1), \quad c = 2$
 - b. $A = (5, 1), \quad c = \frac{1}{2}$
 - c. $A = (3, 4), \quad c = 3$
 - d. $A = (3, 4), \quad c = \frac{1}{2}$
 - e. $A = (3, 4), \quad c = -1$
 - f. $A = (3, 4), \quad c = -2$
 - g. $A = (3, 4), \quad c = -\frac{1}{2}$
2. In problems 1c–1g you took a single point, $(3, 4)$, and scaled it by a bunch of numbers. Try it again with another point, say $B = (4, 6)$. Scale B by eight or ten numbers and plot the resulting points on a single coordinate system. What do you get? Draw a picture of the set of points you'd get if you looked at every possible scalar multiple of B . That is, describe the set of points

$$\{tB \mid t \text{ ranges over all real numbers}\}.$$
3. **Write and Reflect** Suppose A is a point and c is some number. Describe how cA is geometrically related to A . How would you tell someone how to locate $2A$ if that person (but not you) knows where point A is? How about $-3A$? How is cA algebraically related to A ?
4. **Write and Reflect** Why do you think the operation is called "scaling" instead of "multiplying"?

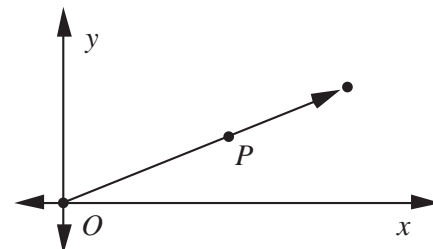
ANOTHER WAY TO DO IT

There is another way to scale points that's useful when you don't want to use coordinates. Here's the idea: Suppose you have a point P and you want to scale it by some number c , such as 2.



Remember, a vector is a directed line segment of a particular length, an arrow with a tail and a head. You'll find out more about vectors in an upcoming investigation.

Draw a vector from O to P and then take that vector and *stretch* it by a factor of c (in our case, a factor of 2). Now you're at cP .



Sometimes, mathematicians use things before they know exactly why they work. And sometimes, *using* things helps you figure out why they work.

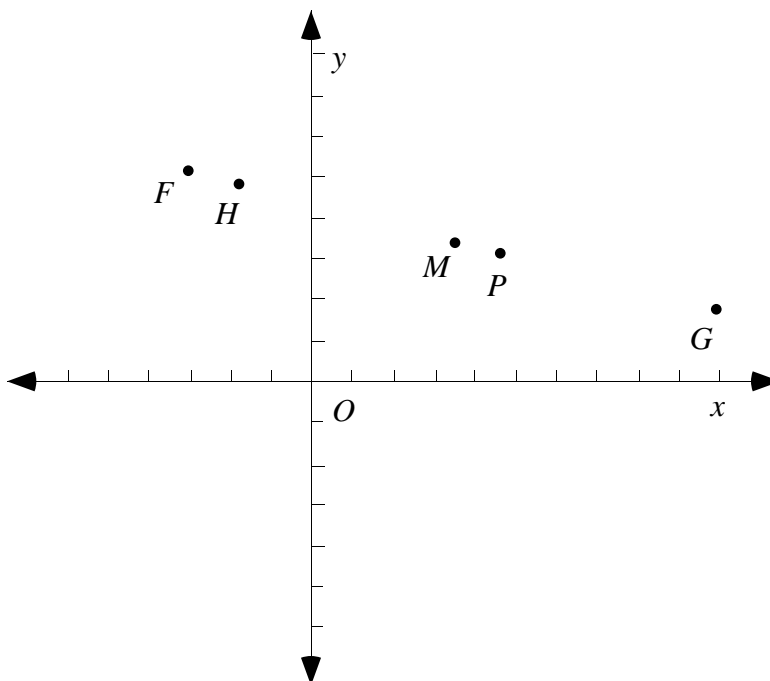
One way to stretch by 2 is to pass a ruler through O and P and to run past P a distance equal to OP . A compass also comes in handy here.

Of course, there are all kinds of things to worry about. What does it mean to “stretch” or to “shrink” a vector? What exactly *is* a vector? *Why* does it work? But these are questions for another day. For now, you might just try to use the technique in some problems. For practice, try this one:

- Suppose $A = (4, 6)$. Use vectors to locate $2A$, $3A$, $\frac{1}{2}A$, $\frac{2}{3}A$, and $-2A$. Check with coordinates.

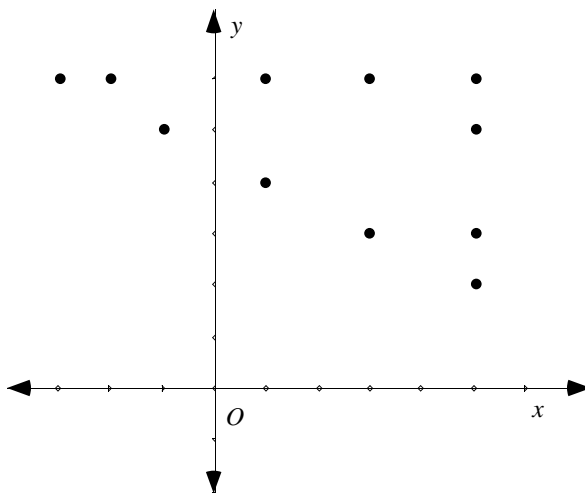
Instead of taking one point and scaling it by a bunch of numbers, what happens if you take a bunch of collinear points and scale them by the *same* number? The next few problems ask you to look at that question.

6. Draw a picture of what you get if each of the five collinear points below is scaled by 2. Describe in words what you get.



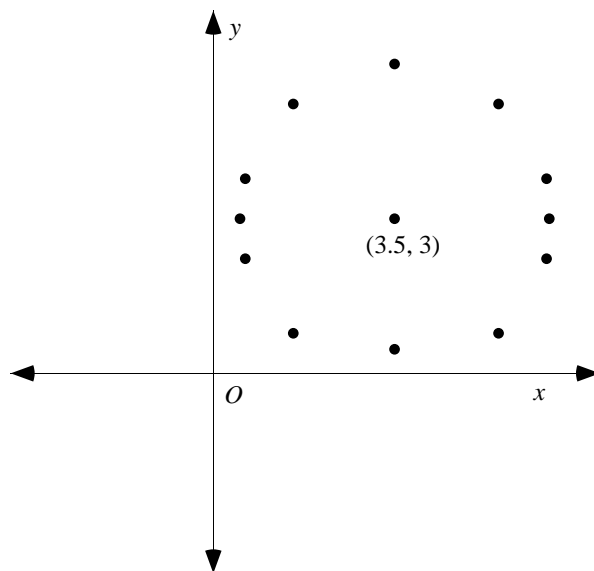
You may want to trace the following pictures on graph paper.

7. Draw a picture of what you get if each of the points below is scaled by $\frac{1}{2}$. Describe in words what you get.



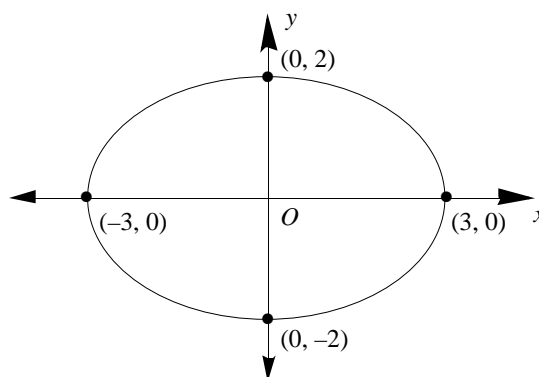
Did you need to scale every point in order to know what you'd get?

8. Draw a picture of what you get if each of the points below is scaled by -4 . Describe in words what you get.

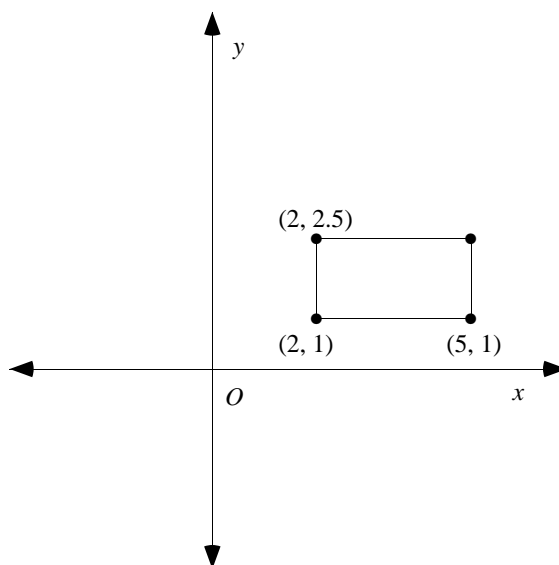


What is the definition of an *ellipse*?

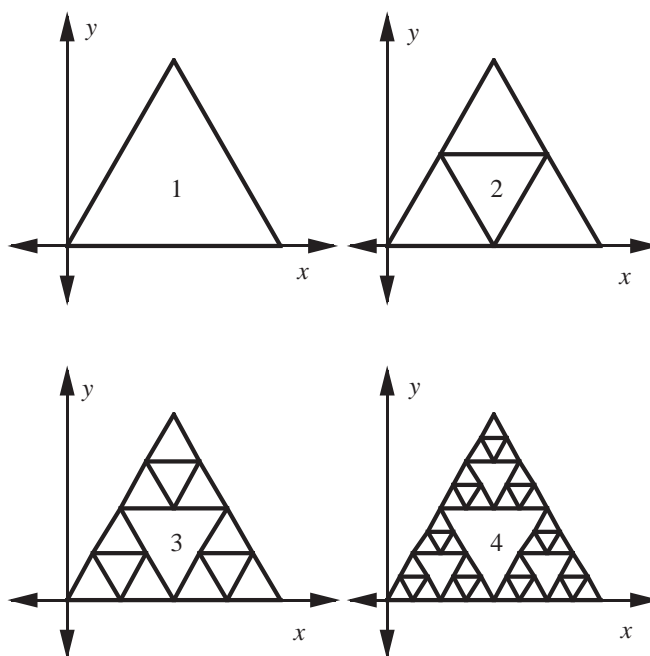
9. Think of a circle centered at the origin of radius 6. What do you get if you scale all the points on the circle by 2? By $\frac{1}{2}$? By $-\frac{1}{2}$?
10. What would you get if you scaled all the points on the ellipse below by $\frac{1}{3}$? By -2 ?



11. Draw what you would get if you scaled every point on the rectangle below by -1 and by $\frac{5}{3}$.



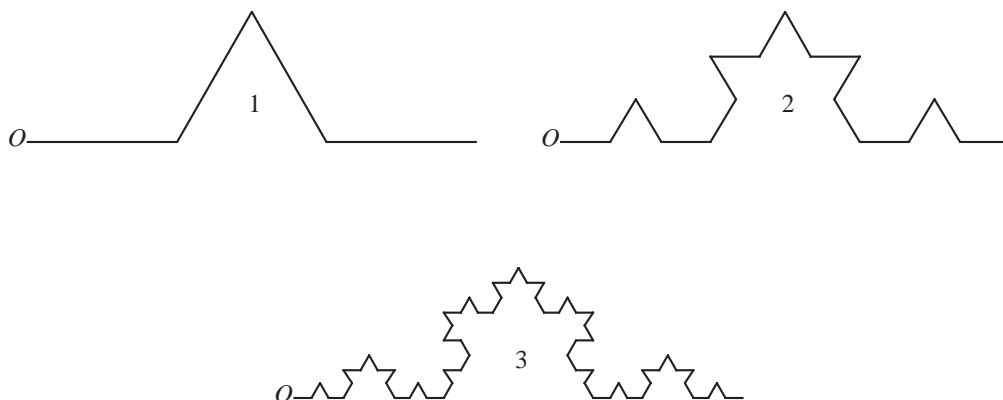
12. Here's a sequence of pictures, each one drawn on a coordinate system.



- a. Describe how each picture is obtained from the previous one. Draw the next one in the sequence.
- b. What happens if you scale each picture by $\frac{1}{2}$?

This is a job for the stretching trick.

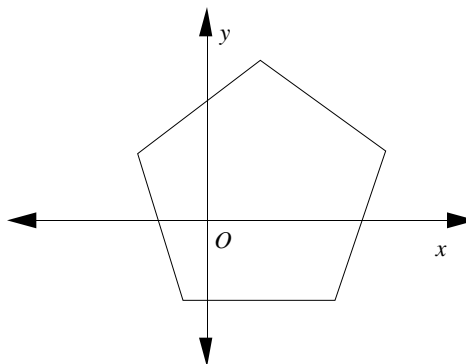
- 13.** Here's a sequence of pictures heading toward a snowflake-like thing.



This kind of repetitive drawing gives you a good feel for how to write a computer program that draws the pictures.

- How is each picture obtained from the previous one? Draw the three pictures (on graph paper, or with a computer), and then draw the next one.
- Place coordinate axes on each picture (with the horizontal axis running along the bottom edge of each snowflake-thing). Describe what you get when you scale each picture by $\frac{1}{3}$.

- 14.** Draw a picture of a pentagon that, if it is scaled by 3, produces this pentagon:



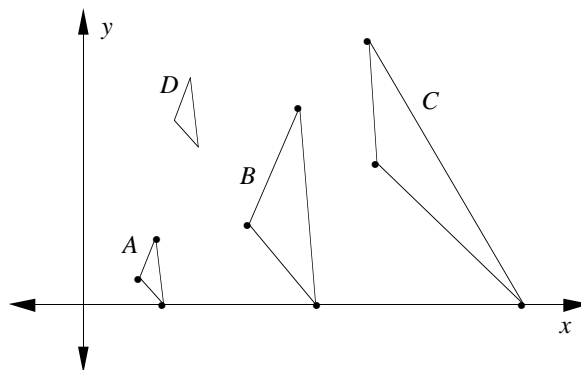
Make sure to note what's necessary about the location of the original pentagon in order to produce the enlarged copy just described.

Problems 16 and 17 each have more than one solution.

15. What would you get if you scaled the pentagon on the previous page by 1?
16. Draw a pentagon that, if scaled by 2 or 3 or $n > 1$, produces an enlarged pentagon, one *side* of which completely contains the corresponding side of the original.
17. Draw a pentagon that, when scaled, has one *vertex* that always stays in the same spot.

CHECKPOINT.....

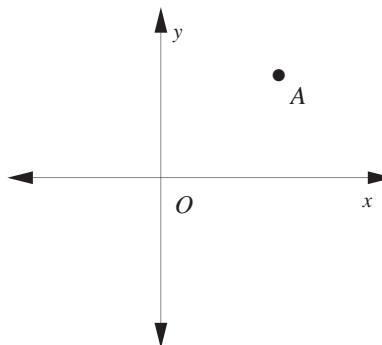
18. Write down everything you know about scaling points. Include the definition. Describe shortcuts you've developed. How do you scale a complicated shape? What happens when you scale by a negative number?
19. There are four triangles below. Which pairs of triangles have the property that one can be scaled to get the other? Justify your answers.



Could a life-size picture of a baby be scaled to match a life-size picture of the adult that the baby becomes?

20. Describe some features that two figures must have in common if it's possible to scale one by a number to get the other one.
21. Suppose $A = (1, 3, 2)$. Use a corner of your classroom to establish a coordinate system and let the unit of measure be 1 foot. Locate A by having someone in the class (with a steady hand) hold a pencil point at A . Now have other people hold pencil points at $2A$, $1.5A$, and $\frac{1}{2}A$. Can someone reach $4A$? Where would $-1A$ be? Give a geometric description of all the points at which people are holding pencil points.

22. True or false: “If the distance from A to B is 10, then the distance from $\frac{1}{2}A$ to $\frac{1}{2}B$ is 5.” If the statement is true, explain or show why. If it’s false, try to fix it.
23. Here’s a picture of point A and the origin O :



- a. Locate $2A$, $\frac{1}{2}A$, and $-2A$.
- b. What can you say about the number c if A is between O and cA ? If cA is between O and A ? If O is between A and cA ?

TAKE IT FURTHER.....

24. True or false? If the statement is true, give a proof or justify it with an argument and include a picture. If false, give a counterexample.
- a. If A is a point and c is a number, then cA is $|c|$ times as far from the origin as A is from the origin.
- b. If C is somewhere on the segment from A to B , then $2C$ is somewhere on the segment between $2A$ and $2B$.
- c. If M is the midpoint of the segment from A to B , then $3M$ is the midpoint of the segment between $3A$ and $3B$.
- d. If the measure of $\angle ABC$ is 45° , then the measure of $\angle(2A)(2B)(2C)$ (the angle that goes from $2A$ to $2B$ to $2C$) is 90° .
25. Suppose $A = (5, 12)$. Prove that $2A$ is collinear with A and the origin.

In Problem 22, you’ve shown this for specific numbers. Use that in a short proof for the general case.

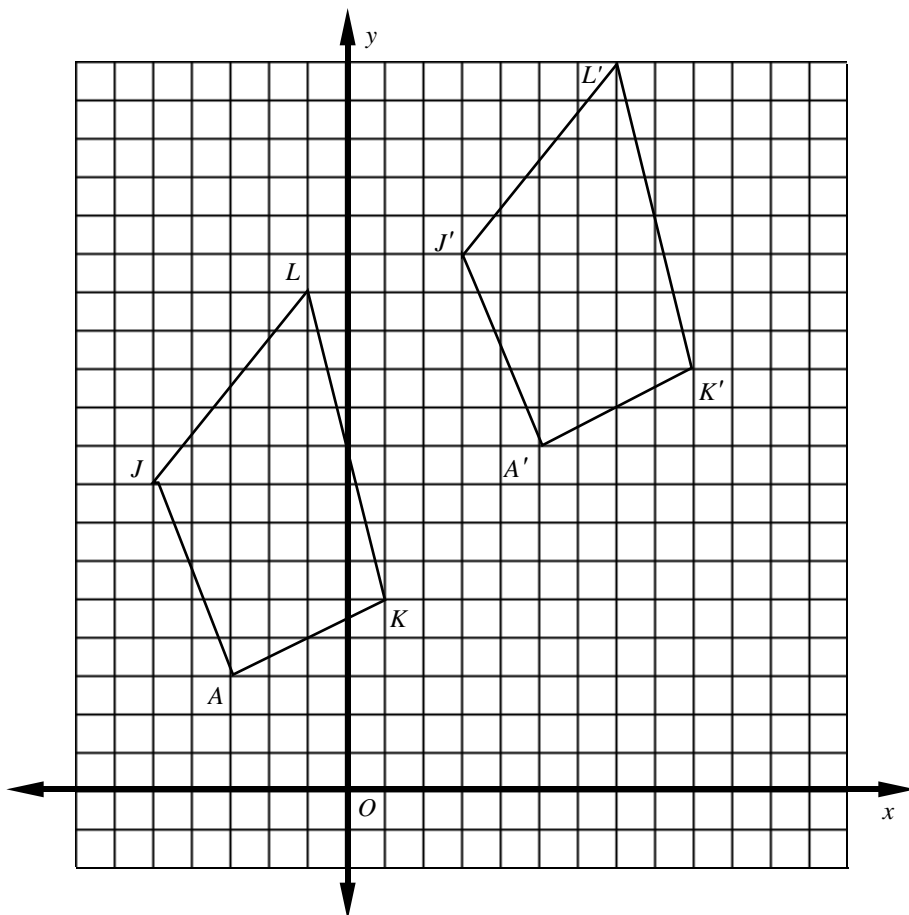
Choose numbers to use for some of these if you need to.

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WAYS TO THINK ABOUT IT

There are several ways to approach this problem. You can use the triangle inequality. Calculate three distances: from O to A , O to $2A$, and A to $2A$. Show that two of the distances add up to the other one. Or, if you did Problems 20–24 of Investigation 5.6, you may be able to use the test for collinearity that involves slope.

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In Problem 5 of Investigation 5.9, you found a rule that would move quadrilateral $AKLJ$ onto $A'K'L'J'$.



Your rule probably looked like this:

$$(x, y) \mapsto (x + 8, y + 6).$$

Another way to write this is

$$(x, y) \mapsto (x, y) + (8, 6).$$

This makes sense. If you walk up to someone and ask, “What’s $(5, 4) + (2, -3)$?” you’ll probably get the response $(7, 1)$, even from someone who makes up the answer on the spot. It turns out that this way of adding ordered pairs is very useful for doing geometry.

Notice that addition is defined in terms of coordinates. In order to add points, we first need to choose a coordinate system.

DEFINITION

To **add two points**, add the corresponding coordinates:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

and

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

For example,

$$(-2, 7) + (4, 1) = (2, 8), \text{ and } (3, 0, 9) + (3, 1, -4) = (6, 1, 5).$$

This allows us to talk about a point as the sum of two other points. We might be doing something, for example, with the three points A , B , and $A + B$.

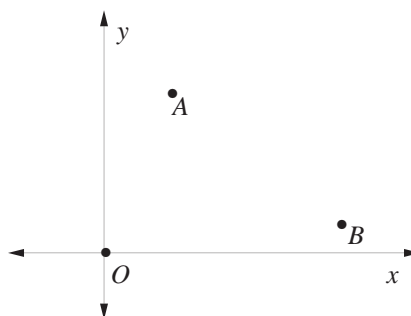
Using this language, Kesia (see Investigation 5.4) could change her way of explaining how to find midpoints. Rather than talking about finding the average of the coordinates, she could talk about finding the average of the points. In referring to the midpoint of two points, she could write it as $M = \frac{1}{2}(A + B)$.

So instead of separating the x - and y -coordinates and figuring out each average independently, she thought of it as adding the two points and averaging the result to get her midpoint.

1. Draw a separate coordinate system for each problem. Then locate O (the origin), A , B , and $A + B$.
 - a. $A = (5, 1)$, $B = (3, 6)$
 - b. $A = (4, -2)$, $B = (0, 6)$
 - c. $A = (4, -2)$, $B = (-3, -5)$
 - d. $A = (4, -2)$, $B = (-4, 2)$
 - e. $A = (4, -2)$, $B = (-1, 4)$
 - f. $A = (3, 1)$, $B = (6, 2)$

Try connecting A , B , O , and $A+B$ in some order.

- g. Describe anything you see in your pictures. How is the location of $A + B$ related to the locations of A , B , and O ? Describe how to locate $A + B$ in this picture:



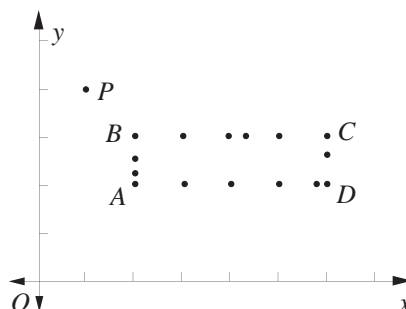
This is the old “when two things are changing, hold one of them constant and let the other change in regular ways” strategy ... otherwise known as the search for invariance.

Another way to ask this is to ask what you’d get if you plotted the set

$$\{A + tB \mid t \in \mathbb{R}\}.$$

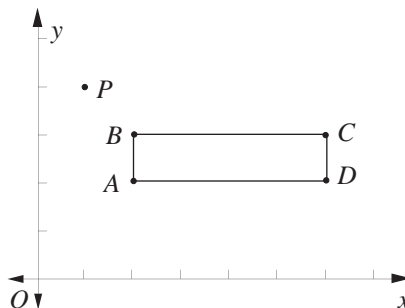
In the previous problem, you took several pairs of points and added them. That was just to give you an idea of what the sum of two points looks like. Now keep one of the points constant and let the second point vary along some familiar sets:

2. Suppose $A = (3, 5)$ and $B = (6, -1)$.
 - a. Plot B , $2B$, $.5B$, $-1B$, $4B$, $-\frac{18}{5}B$, and three or four other multiples of B .
 - b. Add A to each of the points you plotted in part a, and plot the results on the same coordinate system.
 - c. What would the picture look like if you plotted the sum of A and *every possible* multiple of B ?
3. Let $P = (1, 4)$. Plot the sum of P with each of the points on the rectangle below:

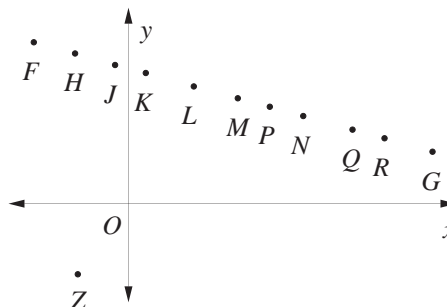


Of course, you can't add P to every point on the rectangle, but Problem 3 gives you an idea of what you'd get if you could.

4. Let $P = (1, 4)$. Plot the sum of P with each of the points on the rectangle below:

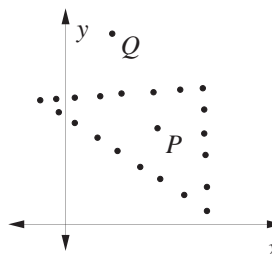


5. Add Z to each of the collinear points and plot the sums.



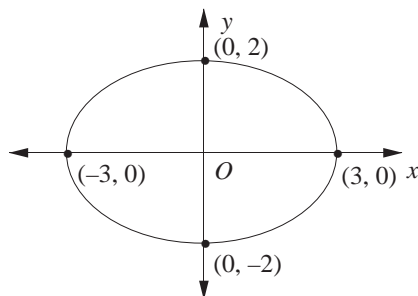
In Problems 5 and 6 you're asked to do something you haven't exactly done before: to add these points without using coordinates at all. Try a different method.

6. Plot the points you get when you add P to each point on the triangle. Do the same for Q .



Draw a picture if it helps to solve the problem or explain the answer.

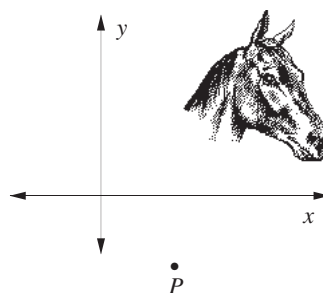
7. Think of a circle of radius 6 centered at the origin. Describe in words what you get if you add all the points on the circle to $(8, -1)$. To $(-4, 2)$? To $(0, 6)$? To $(0, 0)$?
8. Look at this picture:



What would you get if you added $(5, 4)$ to all the points on the ellipse?

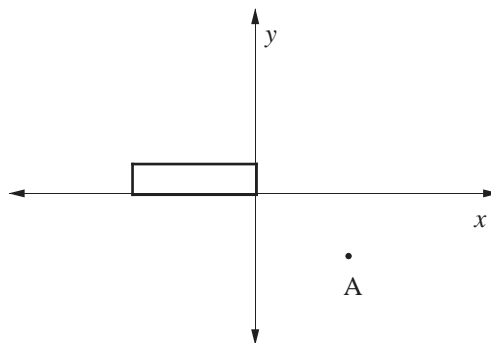
The last few problems ask you to describe the set of points you'd get if you added a point P to *every* possible point on a shape (a circle or an ellipse, for example). You can add P to a *few* points on the shape, and that probably gives you an idea of what to expect. Computers can help you plot many more points than you'd be able to plot by hand.

9. Here's a picture of Trig, drawn on a coordinate system. Sketch the picture that results from adding P to every point on Trig.

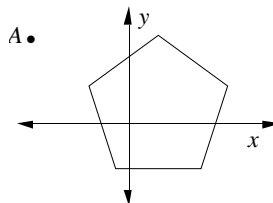


Being able to reverse directions, as you are asked to do in Problems 10 and 11 helps to make your understanding deeper and more flexible.

10. Draw a rectangle such that, if you add A to each of its points, you will get a rectangle like this one:

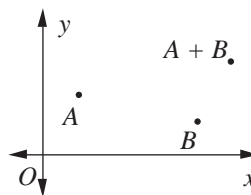


11. Draw a pentagon that, when A is added to every point on it, will produce a pentagon like the one in this picture:



This problem requires access to geometry software.

12. What would you get if you added O to every point on the above pentagon?
13. Design a geometry software sketch that has coordinate axes, two points A and B , and their sum $A + B$. Put a trace on $A + B$.

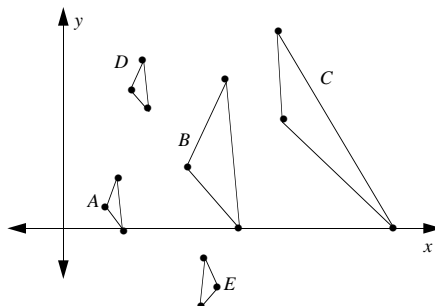


- a. Animate A along a segment or a circle, and find a simple description for the path taken by $A + B$.
- b. Do the same with B .

TYING IT TOGETHER

Take a minute to gather up what you've learned about adding points. If you think hard in answering the next question, you'll solidify what you've learned so far:

14. Write down everything you know about adding points. Include the definition. Describe shortcuts you've developed. What do you get when you add a point to every point on a complicated shape (like a circle, a pentagon, or a picture of Trig)?
15. There are five triangles below. Which pairs of triangles have the property that one can be moved onto the other by adding a point to every point on the triangle? Justify your answers.



16. Describe some features that two figures on a coordinate plane must have in common if it's possible to add a point to every point on one figure to get the other one.
17. Suppose $A = (1, 3, 2)$ and $B = (3, 1, 4)$. Use a corner of your classroom as a coordinate system and let the unit of measure be 1 foot. Locate A by having someone in the class (with a steady hand) hold a pencil point at A . Do the same for B . Now have other people hold pencil points at $B + A$, $B + 2A$, and $B + \frac{1}{2}A$. Can someone reach $B + 3A$? If not, where would it be? Give a geometric description of all the points at which the "other people" (not the people at A or B) are holding pencil points.

- 18.** Suppose $A = (-1, 4)$, and $B = (3, 2)$. Locate the following points on a coordinate system:

- a.** $A + 2B$
- b.** $3A + 2B$
- c.** $-2A + B$
- d.** $-2A + \frac{1}{2}B$
- e.** $4A + \frac{1}{2}B$
- f.** $kA + jB$

You can do this either by drawing a picture and estimating or by setting up some equations.

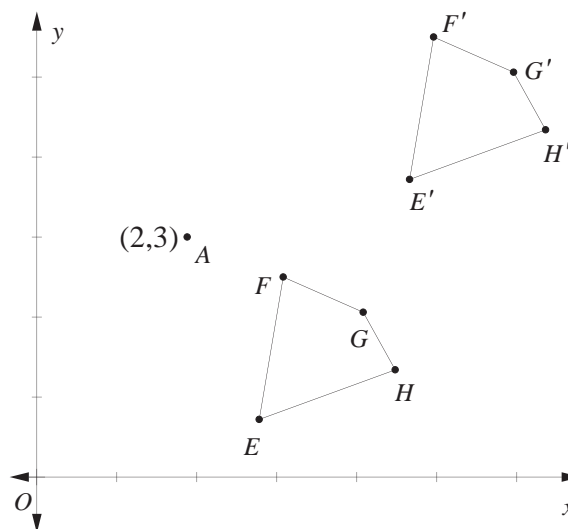
Problems 19 and 20 are really important.

- 19.** Suppose $A = (-1, 4)$, and $B = (3, 2)$ (as in the previous problem).
- a.** Find numbers a and b so that $aA + bB = (-6, 10)$.
 - b.** Can every point on the plane be written as $aA + bB$ for some numbers a and b ? Explain.
- 20.** Suppose $A = (6, 4)$, and $B = (3, 2)$. Can you find numbers a and b so that $aA + bB = (9, 6)$? How about $(-9, 4)$? Describe (with a picture, perhaps) the set of points on the plane that can be written as $aA + bB$ for all possible choices of a and b .

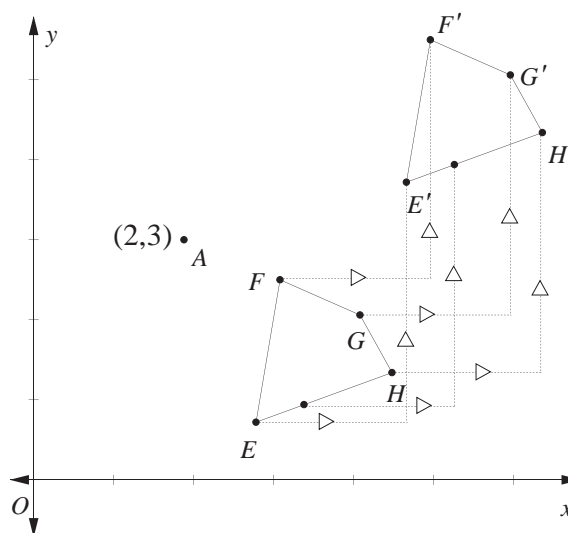
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WAYS TO THINK ABOUT IT

In Investigation 5.11, you looked at a couple of ways to scale points. There are also several ways people think about adding points. Here are a few:

Over and Up What really happens when you add $(2, 3)$ to every point in a figure? Things get pushed to the right 2 units and up 3 units:



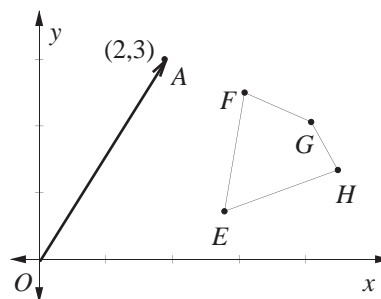
One way to think of this is that the whole quadrilateral is translated to another location by moving first to another spot along the direction of the x -axis and then up or down in the direction of the y -axis to its final location. Every point on the figure is moved over 2 and up 3. The next suggests this by showing what happens to the vertices and one point along one of the sides:



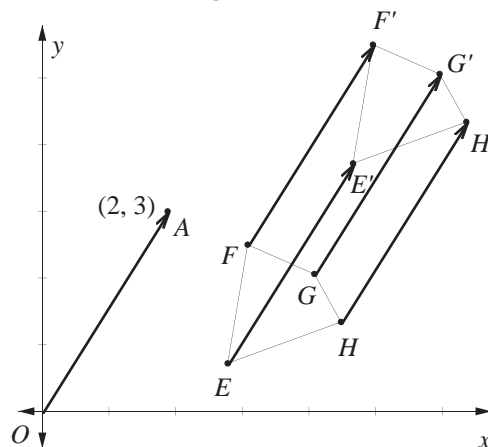
Most geometry software environments have a “translate” or “slide” feature. Many connect this with vectors. Ask your teacher to show you how your particular system works. You can develop a feel for adding points by playing with the software and thinking about what you are doing.

Remember, just think of a vector as an “arrow.” It’s a line segment with a direction and a certain magnitude or length.

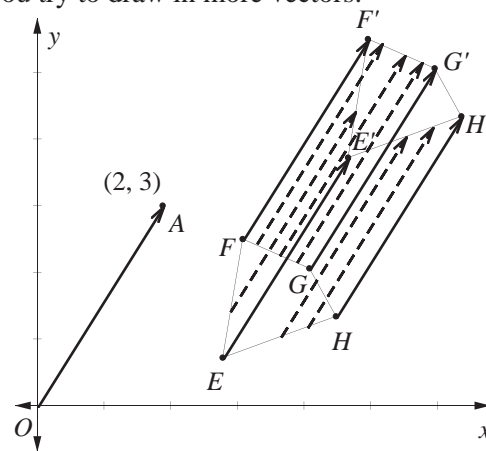
Vectors People also like to think of adding $(2, 3)$ another way. Suppose you want to add $A = (2, 3)$ to every point in $EFGH$. Think of a vector from O to A :



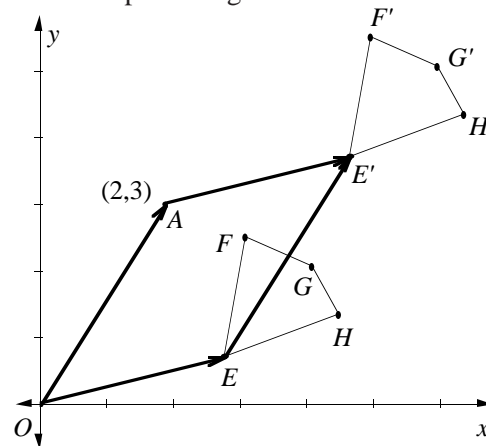
Now imagine that the quadrilateral gets pushed once, this time in the direction of A and as far as the length of A :



Actually, the previous picture shows how only the *vertices* slide along in the direction of A . In fact, every point of $EFGH$ is moved in the direction of A and is moved as far as the length of A , but the picture looks messy if you try to draw in more vectors:



Parallelogram Rule Finally, another way is to concentrate on one point at a time and talk about a “parallelogram rule”:

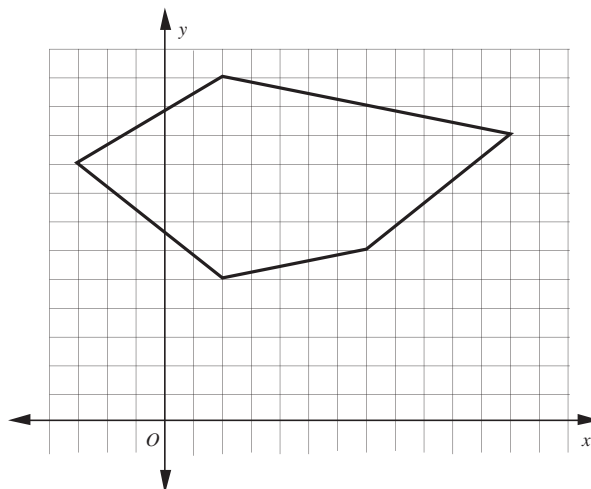


To describe what’s happening, mathematicians say that the quadrilateral has been “translated over 2 and up 3.” Another way to say it is that the quadrilateral has been “translated by $(2, 3)$ ” or even “translated by A .”

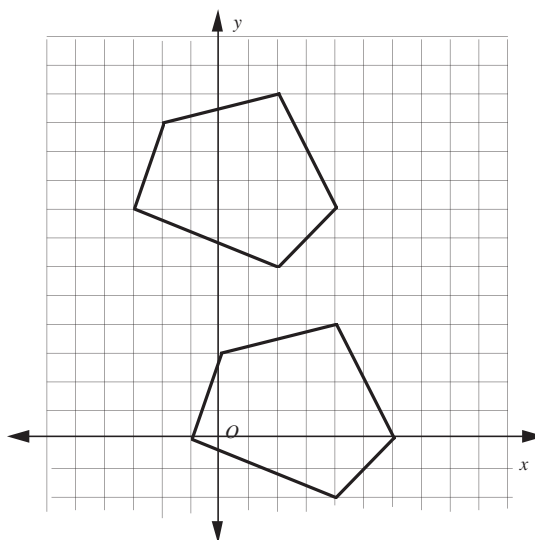
The idea here is that if you want to determine where E ends up when you add A to it, draw the vector OA from O to A (the vector is just the arrow from O to A , not the ray) and the vector OE from O to E . Then $A + E$ (E' in the picture) will be the fourth vertex of the parallelogram, two of whose sides are \vec{OA} and \vec{OE} .

.....

21. Make a copy of the picture below and translate it by $(-2, -4)$.



22. a. By what point should you translate the bottom figure to get the top one?



Did you use the “over and up” vectors or the parallelogram rule?

- b.** By what point should you translate the top figure to get the bottom one?
- c.** Describe the method you used for parts a and b.

MAKING THINGS PRECISE

The setup in this section of the module: We have fixed a coordinate system—an origin, axes, and a unit length, once and for all.

These are two major parts of doing mathematics: solving problems and proving results. Learning mathematics means getting good at both of them.

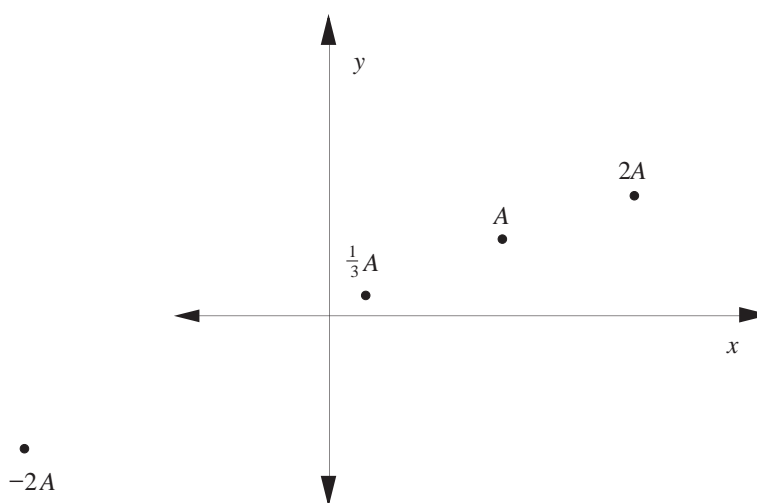
Everything you’ve done in the previous section of the module (Investigations 5.7–5.12) has focused on developing ways to think about how the operations of scaling and adding points behave geometrically. The goal of this investigation is to tie all the loose ends together and to *prove* that what you’ve noticed about scaling and adding points is, in fact, the case. Some of what you’ve seen in the previous investigation can be summarized with phrases like “scaling stretches things,” and “the parallelogram law for adding.” Using a fair bit of algebra and geometry, you can nail these phrases down precisely.

This investigation is a little different from what’s come before. Rather than learn new things now, you need to solidify the foundation for what you’ve learned so far. Learning new things is best accomplished by working on problems; solidifying a foundation is best done by *proving theorems*. Learning to prove theorems is a long process, and one very important part of that process is to read and study existing proofs, all the while asking yourself how the proof’s author might have come up with each of the ideas in the first place. And, just like people learn the lines of a play or a piece of music by heart, it really helps to study a proof so well that you can explain it to someone else. This investigation aims to help you do that with a problem-solving approach to proving theorems.

SCALING

First we’ll investigate what happens to a point when you scale it. Find A in the picture below.

A , $2A$, $\frac{1}{3}A$, and $-2A$ all lie on the same line here. What other familiar point is on that line?



Different scalings of A

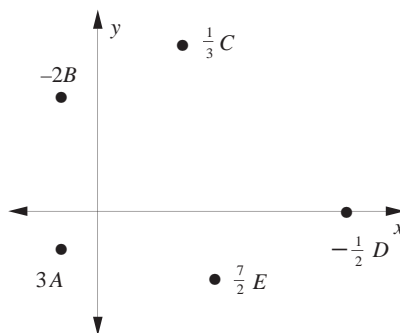
Here's one way to describe what's going on:

- If you scale point A by a number c , one thing is certain: you'll end up on the line that passes through O and A . Where you land depends on c :
 - a. If the value of c is larger than 1, you'll end up at a point beyond A on the ray OA (so that A is between the origin and cA) and cA will be c times as far from O as A is from O .
 - b. If c is between 0 and 1, you'll end up between O and A , again, c times as far from O as A is from O . (Look at $\frac{1}{3}A$ in the picture.)
- If c is negative, you'll end up on the ray opposite to \overrightarrow{OA} . So, O will be between cA and A . And you'll be $|c|$ times as far from O as A is.

What does $|c|$ mean?

1. What happens when $c = 1$? When $c = 0$?
2. Suppose $A = (7, 1)$, $B = (-2, 4)$, and $C = (-3, -6)$. On a coordinate system, draw $\triangle ABC$. Then draw the triangle whose vertices are
 - a. $2A, 2B, 2C$;
 - b. $\frac{1}{3}A, \frac{1}{3}B, \frac{1}{3}C$;
 - c. $-A, -B, -C$.
3. Silly Fran had five points, scaled them, and forgot to keep track of the originals. Help Fran find A, B, C, D , and E .

Use a fresh coordinate system for each problem.
" $-A$ " means $-1A$.



FOR DISCUSSION

Formulate the statement of a theorem that tells precisely how to locate cA with respect to A . Your theorem should account for all the different possible values of c , but should be written in a way that is easy to read and understand.

There are at least three cases to think about:

1. $c \geq 1$
 2. $0 \leq c < 1$
 3. $c < 0$
-

Sometimes when there are several cases, you can find a way to give a single proof that handles all the cases at once.

When there are several cases to consider, it's often a good idea to take one case at a time, and then try to fit everything into a "mega statement" after you've handled the details.

Take one of the cases, say $c \geq 1$ (this is the easiest one). Here's the way the theorem might look for that case:

THEOREM 5.3

Because $c \geq 1$, part 2 of the theorem implies that A is between O and B .

Suppose A is a point, c is a number greater than or equal to 1, and let $B = cA$.

1. B is collinear with A and the origin.
 2. B is c times as far from the origin as A is.
-

The Triangle Inequality says that, if O , A , and B are vertices of a triangle (which happens when the three points are noncollinear), $OA + AB > OB$. When $OA + AB = OB$, then O , A , and B are collinear.

Convention: Single lower-case letters (like c or k) will stand for numbers, and single upper-case letters (like A or X) will be points.

One way to prove the collinearity part of the theorem is to show that $OA + AB = OB$. To prove the second part, just show that $OB = c(OA)$.

FOR DISCUSSION

There are a lot of letters flying around here, and keeping track of what they mean takes a little practice. For example, if A is a point and c is a number, then cA is a *point*, but $c(OA)$ is c times the distance from O to A , so it's a *number*. One way to keep things straight in your head is to substitute numerical values for the letters and see what *kind* of thing you get. For example, to figure out what kind of thing " $c(OA)$ " is, say to yourself, "If $A = (3, 4)$ and $c = 2$, then $OA = 5$ (do what's in parentheses first), so $c(OA) = 10$, a number." Try it. Classify each of these expressions as a "point", a "number", or "meaningless."

1. cB
 2. AB
 3. $c(AB)$
 4. $k(OA)$
 5. $A(BC)$
 6. x
 7. xX
 8. XX
 9. Bc
-

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WAYS TO THINK ABOUT IT

Here is the typical way statements are proved when using coordinates:

- First set up a figure that illustrates what you are trying to prove. Label all the the important points and their coordinates.

- Place the figure on the coordinate system in a position that will simplify the numerical and algebraic calculations. For example, place one vertex of the figure at the origin and, if possible, one of its sides along one of the axes.
- Prove the statement for the specific points, but don't just crunch the numbers: *concentrate on precisely describing to yourself the steps you are taking*. Make the calculation so mechanical that the steps not only work for *these* numbers—they work for *any* numbers.
- Assign *generic* coordinates to the points in your figure. That means to replace things like $A = (3, 4)$ by $A = (x, y)$ or $A = (a_1, a_2)$.
- Use algebraic calculations to prove the statement for your generic setup. Use the mechanical calculations you used in the third item on this list, but do them with letters instead of numbers.
- Then, since the result holds for the generic coordinates, it holds for any numbers you assign the coordinates; that is, it holds for any coordinates whatsoever.

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The following group of problems asks you to implement the third item on the preceding bullet list. It asks you to prove exactly the same statement for several different points. This isn't for the sake of busywork; the most important part of the problems is to construct in your mind a process for proving the statement that works for all the given cases and that will, in fact, work for *any* case.

Can you draw a picture of what you are asked to show?

- 4.** Let $A = (3, 4)$, $c = 3$, and $B = cA$. Show that

$$OA + AB = OB$$

and that $OB = c OA$.

- 5.** Let $A = (-5, 12)$, $c = 2$, and $B = cA$. Show that

$$OA + AB = OB$$

and that $OB = c OA$.

- 6.** Let $A = (6, 8)$, $c = 1.5$, and $B = cA$. Show that

$$OA + AB = OB$$

and that $OB = c OA$.

7. Let $A = (3, 8)$, $c = 4$, and $B = cA$. Show that

$$OA + AB = OB$$

and that $OB = c OA$.

Translate what you want to prove into an algebraic statement; then use algebra to prove the statement.

8. Try proving it for the general case. You could let $A = (x, y)$ or (a_1, a_2) .

Here's a proof of Theorem 5.3 that's probably pretty much like what you did in Problem 8:

Proof of Theorem 5.3. Suppose $A = (a_1, a_2)$, c is a number greater than or equal to 1, and $B = cA$. This means that $B = (ca_1, ca_2)$. There are two things to prove:

Thing 1: $OA + AB = OB$.

$$\begin{aligned} OA + AB &= \sqrt{a_1^2 + a_2^2} + \sqrt{(ca_1 - a_1)^2 + (ca_2 - a_2)^2} \\ &= \sqrt{a_1^2 + a_2^2} + \sqrt{[a_1(c - 1)]^2 + [a_2(c - 1)]^2} \\ &= \sqrt{a_1^2 + a_2^2} + \sqrt{a_1^2(c - 1)^2 + a_2^2(c - 1)^2} \\ &= \sqrt{a_1^2 + a_2^2} + \sqrt{(c - 1)^2(a_1^2 + a_2^2)} \\ &= \sqrt{a_1^2 + a_2^2} + \sqrt{(c - 1)^2} \sqrt{a_1^2 + a_2^2} \\ &= \sqrt{a_1^2 + a_2^2} + (c - 1) \sqrt{a_1^2 + a_2^2} \\ &= (1 + (c - 1)) \sqrt{a_1^2 + a_2^2} \\ &= c \sqrt{a_1^2 + a_2^2} \end{aligned}$$

and

$$\begin{aligned} OB &= \sqrt{(ca_1)^2 + (ca_2)^2} \\ &= \sqrt{c^2 a_1^2 + c^2 a_2^2} \\ &= \sqrt{c^2(a_1^2 + a_2^2)} \\ &= \sqrt{c^2} \sqrt{a_1^2 + a_2^2} \\ &= c \sqrt{a_1^2 + a_2^2}. \end{aligned}$$

So, $OA + AB = OB$, as desired.

9. **Write and Reflect** There are many details that need to be filled in here. Go through the proof, giving reasons for each of the steps.
10. **Write and Reflect** The assumption in the theorem is that $c \geq 1$. Is that assumption ever used in the proof? Trace the above proof with several different numbers; try 3, 2, $\frac{1}{2}$, -1 and -2 . Where does the proof fail for the numbers that are less than 1? Why?

Thing 2: $OB = c OA$. If you look carefully at the proof of Thing 1, you'll see that both OA and OB were calculated in the course of proving $OA + AB = OB$. And it turned out that

$$OA = \sqrt{a_1^2 + a_2^2}$$

and

$$OB = c\sqrt{a_1^2 + a_2^2}.$$

So, by substitution, $OB = c OA$.

Once you do Problem 11, you'll know what scaling by c does for any $c \geq 0$.

- 11.** The proof for Theorem 5.3 assumes that $c \geq 1$. Modify the proof so that it works for $0 \leq c < 1$.
- 12.** Now you know that Theorem 5.3 works for any $c \geq 0$. Prove that it also holds for negative c . *Hint:* If $c < 0$, then $-c > 0$. How is $-cA$ related to cA ?

FOR DISCUSSION

Can you prove your megatheorem with one proof that handles all cases?

Theorem 5.3 is still worded in a way that assumes that $c \geq 1$. But, in Problems 11 and 12, you described what scaling does for any kind of c . As a class, state a new and improved version of Theorem 5.3 that tells the whole story in one or two sentences.

ADDING POINTS

Now for adding. If A and B are points, we want a theorem that tells us how to locate $A + B$ in terms of A and B . A little experimenting is in order:

- 13.** Plot each pair of points as well as their sum. Describe how the sum is located with respect to the points and the origin.
 - a.** $A = (-3, 5)$, $B = (5, 1)$.
 - b.** $A = (2, 3)$, $B = (5, -1)$.
 - c.** $A = (-2, -2)$, $B = (0, 6)$.

d. $A = (-3, 3), B = (5, -5).$

e. $A = (-3, 5), B = 2(5, 1).$

f. $A = (-3, 5), B = \frac{1}{2}(5, 1).$

14. In the previous problem, draw segments connecting the origin, O , to A and B , and segments connecting $A + B$ to A and B . What kind of figure do you get in each case?

The last two problems suggest a theorem:

THEOREM 5.4

If $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then $A + B$ is the fourth vertex of the parallelogram that has A , O , and B as three of its vertices and \overline{OA} and \overline{OB} as two of its sides.

15. This seems like a pretty wordy way of stating a theorem. Do you really need the “and \overline{OA} and \overline{OB} as two of its sides” part? Try to make the theorem clearer by stating it in your own words. Use a picture if you think it helps.

Now for the proof. Basically, you want to show that the quadrilateral whose vertices are O , A , $A + B$, and B is a parallelogram. The strategy is *to show that the opposite sides have the same length*.

This makes use of an important theorem: “A quadrilateral is a parallelogram if its opposite sides have the same length.”

..... WAYS TO THINK ABOUT IT

Once again, the proof will use *generic* coordinates; it will translate the thing to be proved into a statement about coordinates and then use a little algebra.

And again, the best way to figure out how to prove a statement about generic points is to prove it for specific points and concentrate on precisely describing to yourself the steps involved.

The next group of problems asks you to prove exactly the same statement for several different points. This isn't for the sake of busywork; the most important part of these problems is to construct in your mind a process for proving the statement that works for all the given cases and that will, in fact, work for *any* case.

.....

Can you draw a picture of what you are asked to show?

Ask your teacher to show you how to do this in a way that doesn't make use of anything special about the specific numbers in the problem.

The previous problem set was designed to help you identify the key calculations.

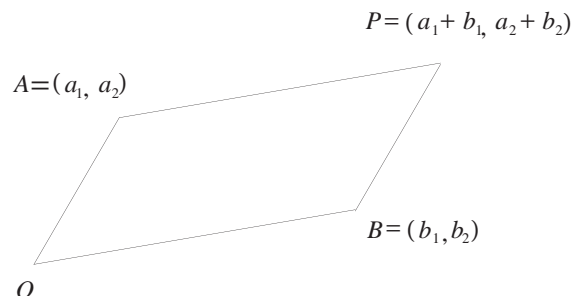
16. Let $A = (3, 4)$, $B = (5, -12)$, and let $P = A + B$. Draw a picture of O , A , B , and P . Show that $OA = BP$ and $OB = AP$.
17. Let $A = (8, 15)$, $B = (-4, 3)$, and let $P = A + B$. Draw a picture of O , A , B , and P . Show that $OA = BP$ and $OB = AP$.
18. Let $A = (8, 6)$, $B = (3, 1)$, and let $P = A + B$. Draw a picture of O , A , B , and P . Show that $OA = BP$ and $OB = AP$.
19. Here's your chance to try doing the proof for the general case. Use $A = (a_1, a_2)$ and $B = (b_1, b_2)$.

Here's a proof of Theorem 5.4 that captures what's common to all the calculations in the problem set above.

Trial proof of Theorem 5.4. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$. Let $P = A + B$, so that

$$P = (a_1 + b_1, a_2 + b_2).$$

The situation looks like this:



Actually, we *hope* it looks like this. We're trying to show that it is a parallelogram.

To prove the opposite sides have the same length, you need to show two things:

Thing 1: $OA = PB$.

We have

$$OA = \sqrt{a_1^2 + a_2^2}$$

and

$$\begin{aligned} PB &= \sqrt{[(a_1 + b_1) - b_1]^2 + [(a_2 + b_2) - b_2]^2} \\ &= \sqrt{a_1^2 + a_2^2}, \end{aligned}$$

so $OA = PB$.

Thing 2: $OB = PA$.

20. Prove Thing 2.

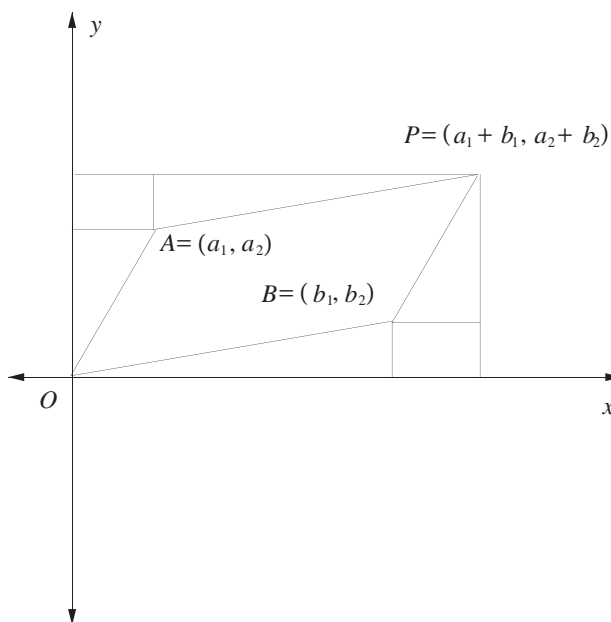
21. Does Theorem 5.4 hold if O , A , and B are collinear? For example, where is the parallelogram if $A = (9, -1)$ and $B = (18, -2)$?

22. If A , B , and C are points, show that the quadrilateral whose vertices are B , C , $C + A$, and $B + A$ is a parallelogram by showing that the opposite sides are congruent. Draw a picture.

23. Don't get nervous, but there's actually a glitch in our proof of Theorem 5.4. We'll discuss it in Investigation 5.17, but can you find it? *Hint:* Can $OA = PB$ and $OB = PA$ for some other point P that *doesn't* complete the parallelogram?

The segment from $B + A$ to $C + A$ is opposite \overline{BC} , and is obtained from \overline{BC} by "translating by the vector from O to A ."

- 24. Write and Reflect** There's another way to think about Theorem 5.4 that might avoid the glitch in Problem 23.



- Label lengths in the figure above. Use congruent triangles and corresponding parts to show that the opposite sides of $OBPA$ are parallel.
- Draw a picture that shows the situation if A is in the second quadrant. Develop a proof for this case.
- How many cases would you need to claim a complete proof?

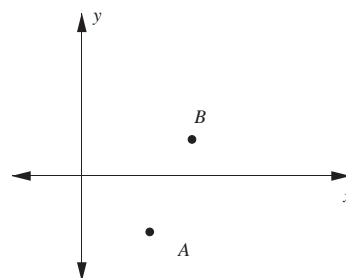
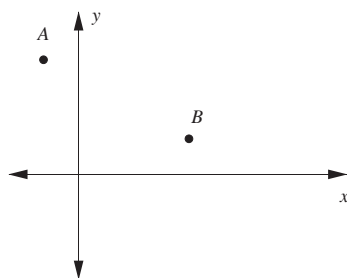
USING THE THEOREMS

You now have a pair of theorems that tells you exactly what happens when you add and scale points. Scaling points by c stretches or shrinks by a factor of $|c|$ (and changes direction if $c < 0$), and adding points “completes” a parallelogram. This is pretty nice.

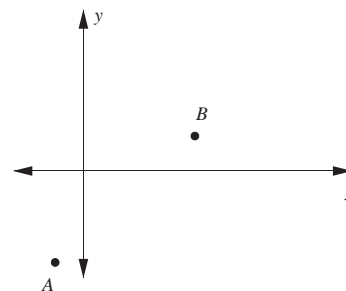
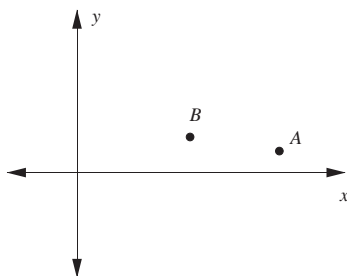
And it’s pretty useful, too. It takes a little practice to think this way, but once you get in the habit of using these two theorems, you will be able to solve all kinds of problems. The next set of exercises will help you get in the right mood for applying the theorems:

What is $-1A + A$?

1. Explain how to locate $-1A$ in the plane if you know A .
2. Copy each of the pictures below, and locate $B - A$. Find a way to describe the position of $B - A$ in terms of the origin, A , and B .



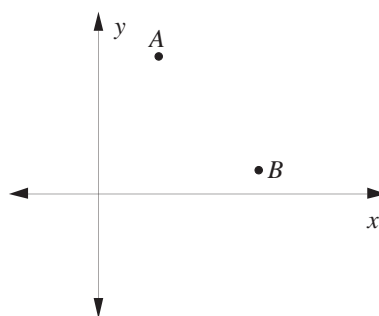
$B - A$ means $B + (-1A)$. One way to find it is to find $-1A$ and then “complete the parallelogram” between $-1A$ and B .



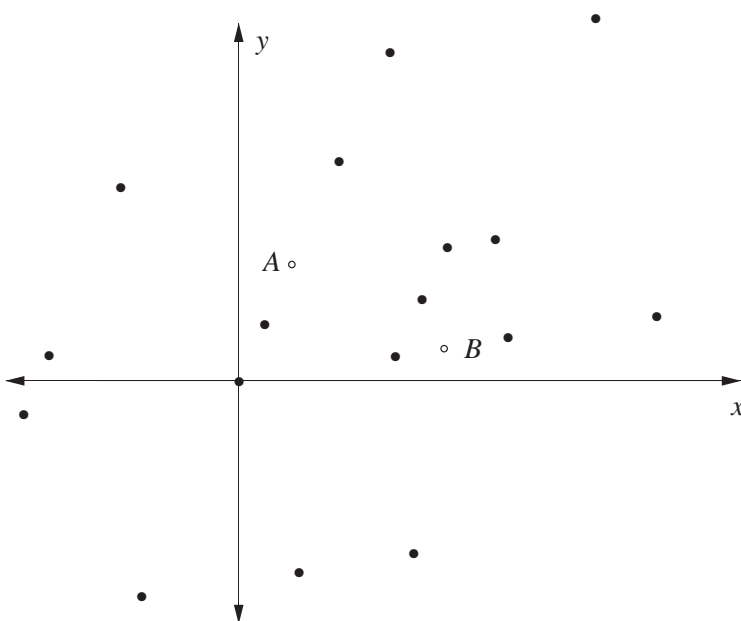
Think of $2A + 3B$ as “two A s and three B s.” Go out two A s from the origin, go out three B s from the origin, and complete the parallelogram.

3. Copy the picture below and locate the following points:

- a. $2A + 3B$;
- b. $3A - B$;
- c. $-2A - \frac{1}{2}B$;
- d. $.7A + 3.6B$.



4. Points A and B are marked with open circles. For each of the other points in the picture, estimate numbers x and y so that $x A + y B$ will get you to the point.

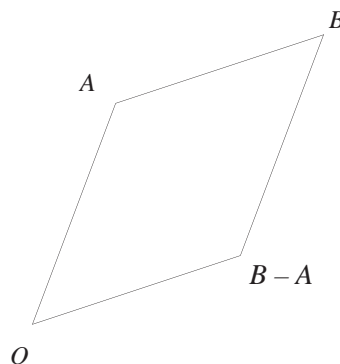
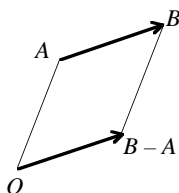


Here's a useful theorem that follows from Theorem 5.4.

THEOREM 5.5

If A and B are points, then the segment from O to $B - A$ is parallel and congruent to \overline{AB} .

One way to think about this is that the vector from A to B is "equivalent" to the vector from O to $B - A$.



5. Show that Theorem 5.5 holds if $A = (3, 5)$ and $B = (7, 9)$.

Here is a proof of Theorem 5.5: Since

$$A + (B - A) = B,$$

point B completes the parallelogram whose vertices are O , A , and $B - A$, and whose sides are \overline{OA} and the segment from O to $B - A$. Since the opposite sides of a parallelogram are congruent and parallel, the segment from O to $B - A$ is parallel and congruent to \overline{AB} .

6. Show that if P and A are points and t is a number, the segment from P to $P + tA$ is parallel to the segment from O to A . If you get stuck, try it with actual coordinates. Even if you don't get stuck, draw a picture.
7. Suppose $A = (4, 1)$, $B = (9, 5)$, $C = (3, -1)$, and $D = (13, 7)$. Show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

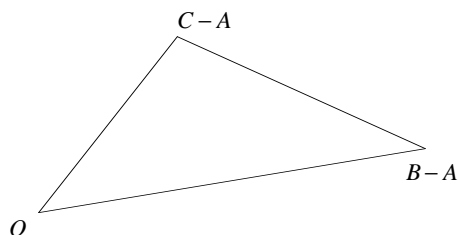
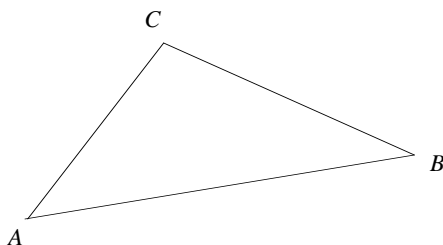
8. Suppose that A, B, C and D are points and t is a number so that $B - A = t(D - C)$. Show that $\overrightarrow{AB} \parallel \overrightarrow{CD}$. If you get stuck, try it with actual coordinates. Even if you don't get stuck, draw a picture.

9. If A and B are points, show that the midpoint of \overline{AB} is

$$\frac{1}{2}(A + B).$$

If you get stuck, try it with actual coordinates. Even if you don't get stuck, draw a picture. See the discussion on page 85.

10. Suppose P and A are points. If A is the midpoint of \overline{PX} , express X in terms of P and A .
11. Suppose A is a point. If O is the midpoint of \overline{AC} , express C in terms of A .
12. Let $A = (4, 4)$, $B = (8, -4)$ and look at $\triangle OAB$. Let M be the midpoint of \overline{OA} and N be the midpoint of \overline{OB} . Show that $\overline{MN} \parallel \overline{AB}$ and $MN = \frac{1}{2}AB$.
13. If $A = (3, 2)$, $B = (-4, 5)$, and $C = (7, 4)$, show that $\triangle ABC \cong \triangle O(B - A)(C - A)$. Draw a picture.
14. If A, B , and C are any three points, show that $\triangle ABC \cong \triangle O(B - A)(C - A)$.



Move it down to O .

This is just the Midline Theorem.

Notice that $O = A - A$.

Sometimes people say that “ $\triangle ABC$ has been translated to the origin by moving A to O .”

Put them all on the same set of axes for a nice effect.

- 15. Continuation** Draw three points A , B , and C . Draw the pictures of the triangles you get by moving
- a.** A to the origin;
 - b.** B to the origin;
 - c.** C to the origin.
- 16.** If A , B , and C are points, show that the quadrilateral whose vertices are B , C , $C + A$, and $B + A$ is a parallelogram by showing that the opposite sides are parallel. Draw a picture.
- 17.** Given two points A and B , how can you find a point that is one third of the way from A to B ? Explain.

THE ALGEBRA OF POINTS

Is it obvious to you that “algebra with points” behaves like algebra with numbers? There are many properties of point addition and scaling that look like familiar number properties. For example, here are eight properties that are often used in a branch of mathematics called *linear algebra*. They are given here in a theorem.

THEOREM 5.6

These statements look like statements about numbers, but they are really about points. Your teacher will help you navigate the fine points of this theorem.

If $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ are points and d and e are numbers, then

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A + O = A$
4. $A + (-1A) = O$
5. $(d + e)A = dA + eA$
6. $d(A + B) = dA + dB$
7. $d(eA) = (de)A$
8. $1A = A$.

1. Prove each part of the theorem.
2. Use adding and scaling points to show that the line joining the midpoints of two sides of a triangle is parallel to the third side and half as long.
3. Suppose A and B are points. Explain how to locate each of these points:

- a. $\frac{1}{3}A + \frac{2}{3}B$
- b. $\frac{2}{3}A + \frac{1}{3}B$
- c. $\frac{1}{4}A + \frac{3}{4}B$
- d. $\frac{3}{4}A + \frac{1}{4}B$

If you get stuck, try it with actual coordinates. Even if you don't get stuck, draw a picture.

e. $\frac{3}{5}A + \frac{2}{5}B$

f. $kA + (1 - k)B$ (here, $0 \leq k \leq 1$).

CHECKPOINT.....

4. Let $A = (4, 2)$, $B = (5, -3)$, $C = (6, 4)$, and

$$P = \frac{1}{3}(A + B + C).$$

Show that, in $\triangle ABC$, P is $\frac{2}{3}$ of the way from any vertex to the midpoint of the opposite side.

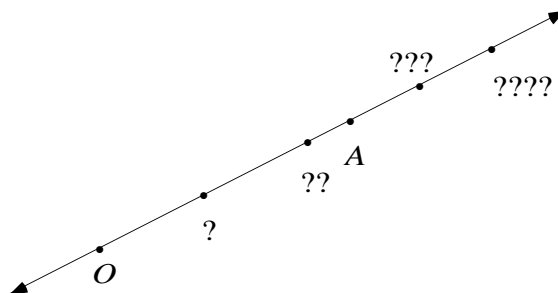
TAKE IT FURTHER.....

5. Let A , B , and C be points, and let

$$P = \frac{1}{3}(A + B + C).$$

Show that, in $\triangle ABC$, P is $\frac{2}{3}$ of the way from any vertex to the midpoint of the opposite side.

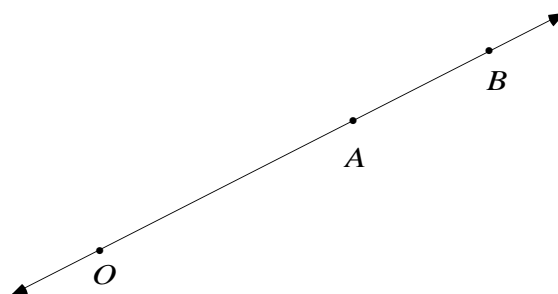
Theorem 5.3 tells you what happens if you scale a point by a number. If $B = cA$, then B is collinear with A and O . But is *every* point on \overleftrightarrow{OA} a multiple of A ?



Are they all multiples of A ?

Yes indeed.

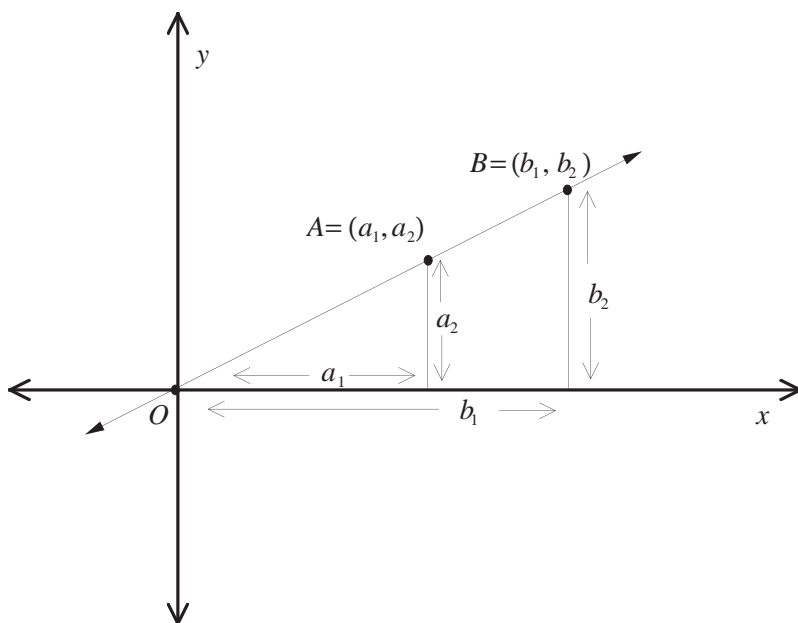
The proof makes use of what you know about similar triangles. Just as in the proof of Theorem 5.3, you have to worry about where B is located in relation to the origin and A . Look first at the special case where A is between the origin and B :



Is B a multiple of A ?

As usual, assign generic coordinates to A and B . Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$.

Then, thinking about a coordinate system, the measurements look like this:



There are two right triangles in the picture (what are they?) and, *because* O , A , and B are collinear, the triangles are similar. Since corresponding sides of similar triangles are proportional,

$$\frac{b_2}{a_2} = \frac{b_1}{a_1}.$$

Give this common ratio a name; call it “ c .” So,

$$\frac{b_2}{a_2} = \frac{b_1}{a_1} = c.$$

An important habit of mind: Give things names so you can more easily deal with them.

But then, $b_2 = ca_2$ and $b_1 = ca_1$ (why?), so

$$B = (b_1, b_2) = (ca_1, ca_2) = c(a_1, a_2) = cA,$$

and that’s what was to be proved. The result can be summarized in a theorem:

THEOREM 5.7

Suppose A and B are points so that B is collinear with A and the origin. Then there is a number c so that $B = cA$.

1. Write out the proof of Theorem 5.7 in your own words and explain it to a friend.
2. Prove Theorem 5.7 in the case where B is between O and A .
3. Prove Theorem 5.7 in the case where O is between B and A .
4. Suppose \overleftrightarrow{AB} and \overleftrightarrow{CD} are parallel lines: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Show that there is a number k so that $B - A = k(D - C)$. If you get stuck, try it with actual coordinates. Even if you don't get stuck, draw a picture.

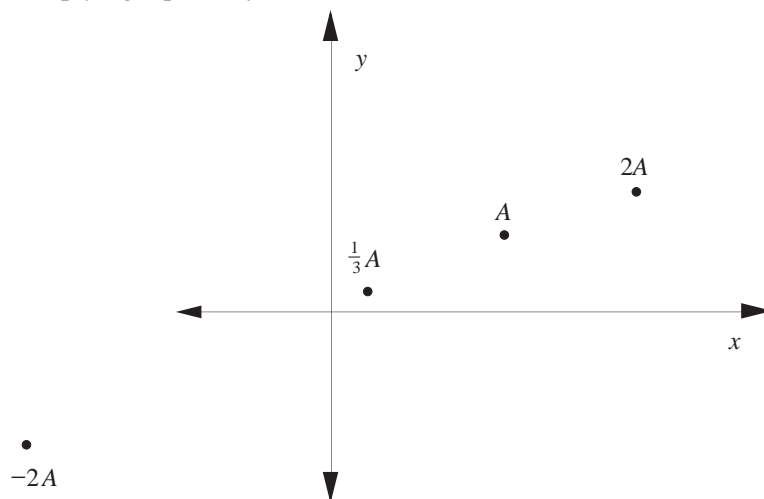
Theorems 5.3 and 5.7 can be combined to give a complete picture:

THEOREM 5.8

If A is a point, the set of all multiples of A is the line through the origin and A .

ONE PROOF FITS ALL

In the “For Discussion” on page 103, you formulated one statement that described the effect of multiplying a point by a number in all cases:



Different scalings of A

The purpose of this section is to help you develop one *proof* that will handle all the cases. What is it that you want to prove? How about something like this:

THEOREM 5.9 (More general form of Theorem 5.3)

Suppose A is a point, c is a number; let $B = cA$.

1. B is collinear with A and the origin.
 2. B is $|c|$ times as far from the origin as A is.
 3. In addition,
 - a. If $c \geq 1$, A is between O and B .
 - b. If $0 \leq c < 1$, B is between O and A .
 - c. If $c < 0$, O is between B and A .
-

Do you remember the proof of Theorem 5.3? It revolved around two calculations:

1.

$$\begin{aligned}
 OA + AB &= \sqrt{a_1^2 + a_2^2} + \sqrt{(ca_1 - a_1)^2 + (ca_2 - a_2)^2} \\
 &= \sqrt{a_1^2 + a_2^2} + \sqrt{[a_1(c - 1)]^2 + [a_2(c - 1)]^2} \\
 &= \sqrt{a_1^2 + a_2^2} + \sqrt{a_1^2(c - 1)^2 + a_2^2(c - 1)^2} \\
 &= \sqrt{a_1^2 + a_2^2} + \sqrt{(c - 1)^2(a_1^2 + a_2^2)} \\
 &= \sqrt{a_1^2 + a_2^2} + \sqrt{(c - 1)^2} \sqrt{a_1^2 + a_2^2} \\
 &= \sqrt{a_1^2 + a_2^2} + (c - 1) \sqrt{a_1^2 + a_2^2} \\
 &= [1 + (c - 1)] \sqrt{a_1^2 + a_2^2} \\
 &= c \sqrt{a_1^2 + a_2^2}
 \end{aligned}$$

2.

$$\begin{aligned}
 OB &= \sqrt{(ca_1)^2 + (ca_2)^2} \\
 &= \sqrt{c^2a_1^2 + c^2a_2^2} \\
 &= \sqrt{c^2(a_1^2 + a_2^2)} \\
 &= \sqrt{c^2}\sqrt{a_1^2 + a_2^2} \\
 &= c\sqrt{a_1^2 + a_2^2}
 \end{aligned}$$

They are more than suspicious; for certain values of c , they are just plain incorrect.

If you did Problem 10, you know that there are two lines here that are suspicious. They are

From item 1,

$$\begin{aligned}
 \sqrt{a_1^2 + a_2^2} + \sqrt{(c-1)^2}\sqrt{a_1^2 + a_2^2} \\
 = \sqrt{a_1^2 + a_2^2} + (c-1)\sqrt{a_1^2 + a_2^2}.
 \end{aligned}$$

From item 2,

$$\sqrt{c^2}\sqrt{a_1^2 + a_2^2} = c\sqrt{a_1^2 + a_2^2}.$$

In the first case, $\sqrt{(c-1)^2}$ is replaced with $c-1$, and in the second case, $\sqrt{c^2}$ is replaced by c . In general, you can't do that. In general,

$$\sqrt{(\text{something})^2} \neq \text{that thing}.$$

For example, if something = -3 , then

$$\sqrt{(-3)^2} = \sqrt{9} = 3,$$

so $\sqrt{(-3)^2} \neq -3$. In general, if you square a *negative* number and take the square root of what you get, you *don't* get back your negative number. What do you get back?

It depends. Because the $\sqrt{\quad}$ function always returns a positive number or zero, “squaring and square-rooting” has the effect of leaving positive numbers alone and turning negative numbers into their opposites. There's *another* operation from algebra that does exactly the same thing. It's the *absolute value* operation. In fact, sometimes the absolute value function is defined like this:

$$|x| = \begin{cases} x & \text{if } x \text{ is nonnegative} \\ -x & \text{if } x \text{ is negative.} \end{cases}$$

The symbol $-x$ doesn't always stand for a negative number. If x is negative, $-x$ is positive. It's the same distance from 0 as $-x$ but on the positive half of the number line.

Why do you think computer programmers like this method for calculating absolute value?

The second line of this definition looks weird. How can $|x| = -x$? Try it with $x = -5$. It says

$$|-5| = -(-5),$$

and that's exactly right.

Another way to calculate absolute value, a favorite of computer programmers, is to take advantage of the fact that “squaring then square-rooting” does exactly the same thing as taking absolute value:

$$\sqrt{x^2} = |x|.$$

5. Looking back at the suspect lines in the proof of Theorem 5.3:

- $\sqrt{a_1^2 + a_2^2} + \sqrt{(c-1)^2} \sqrt{a_1^2 + a_2^2} = \sqrt{a_1^2 + a_2^2} + (c-1) \sqrt{a_1^2 + a_2^2}$
- $\sqrt{c^2} \sqrt{a_1^2 + a_2^2} = c \sqrt{a_1^2 + a_2^2}$

a. For what values of c is it true that

$$\sqrt{(c-1)^2} = (c-1)?$$

Write a more general statement that is true for *all* values of c .

b. For what values of c is it true that

$$\sqrt{c^2} = c?$$

Write a more general statement that is true for *all* values of c .

One way to think about proving Theorem 5.9 is to calculate all the pieces separately. Recall the setup:

$A = (a_1, a_2)$ is a point, c is a number, and let $B = cA = (ca_1, ca_2)$.

The theorem claims that

1. B is collinear with A and the origin.
2. B is $|c|$ times as far from the origin as A is.
3. In addition,
 - a. If $c \geq 1$, A is between O and B ;
 - b. If $0 \leq c < 1$, B is between O and A ;
 - c. If $c < 0$, O is between B and A .

Using what you now know about square roots and absolute values, you can write:

$$\begin{aligned}
 OA &= \sqrt{a_1^2 + a_2^2} \\
 AB &= \sqrt{(ca_1 - a_1)^2 + (ca_2 - a_2)^2} \\
 &= \sqrt{[a_1(c - 1)]^2 + [a_2(c - 1)]^2} \\
 &= \sqrt{a_1^2(c - 1)^2 + a_2^2(c - 1)^2} \\
 &= \sqrt{(c - 1)^2(a_1^2 + a_2^2)} \\
 &= \sqrt{(c - 1)^2} \sqrt{a_1^2 + a_2^2} \\
 &= |c - 1| \sqrt{a_1^2 + a_2^2},
 \end{aligned}$$

and

$$\begin{aligned}
 OB &= \sqrt{(ca_1)^2 + (ca_2)^2} \\
 &= \sqrt{c^2 a_1^2 + c^2 a_2^2} \\
 &= \sqrt{c^2(a_1^2 + a_2^2)} \\
 &= \sqrt{c^2} \sqrt{a_1^2 + a_2^2} \\
 &= |c| \sqrt{a_1^2 + a_2^2}.
 \end{aligned}$$

We're using the triangle equality three times to prove Theorem 5.3.

- 6.** Using these results, modify the proof of Theorem 5.3 in Investigation 5.13 (see pages 102–103) to show that
 - a.** If $c \geq 1$, $OA + AB = OB$.
 - b.** If $0 \leq c < 1$, $OB + BA = OA$.
 - c.** If $c < 0$, $BO + OA = BA$.

TAKE IT FURTHER.....

- 7.** Extend the results of this investigation to three dimensions. In particular, how do Theorems 5.7 and 5.9 play out in three dimensions?

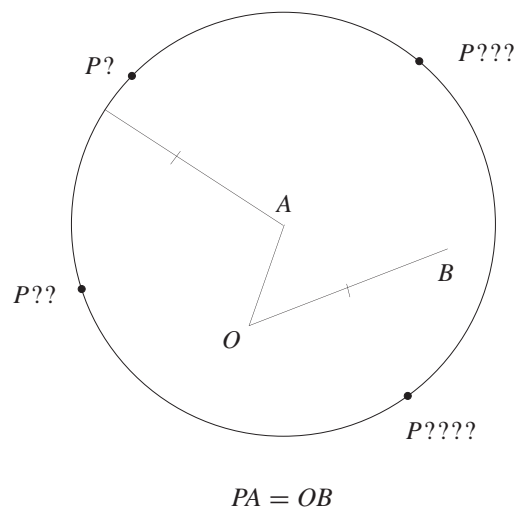
MORE ON ADDING POINTS

Do you remember Theorem 5.4? It shows how to locate $A + B$ in terms of A and B . If $P = A + B$, then P is the fourth vertex of a parallelogram whose sides are \overline{OA} and \overline{OB} . Now, there's only one fourth vertex of a parallelogram whose sides are \overline{OA} and \overline{OB} , so we don't need to worry about other points fitting the bill. This seems simple and foolproof.

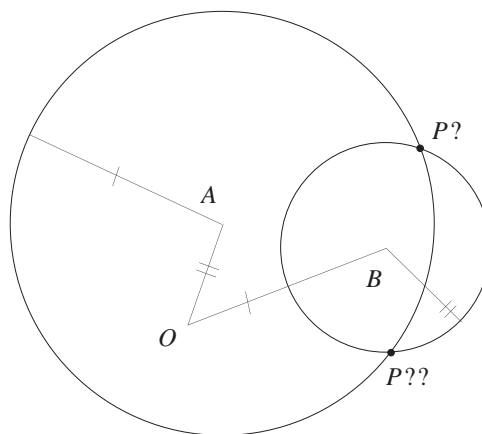
But there's a glitch. What you actually proved was that $PA = OB$ and $PB = OA$, reasoning that that would make $OBPA$ a parallelogram because its opposite sides would be congruent. But could it happen that $PA = OB$ and $PB = OA$ *without* $OBPA$ being a parallelogram?

Well, yes.

Think of it this way: $PA = OB$ puts P on a circle with center A and radius OB :

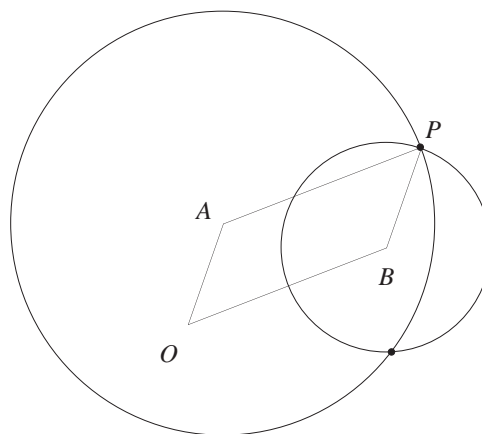


And $PB = OA$ puts P on another circle with center B and radius OA :



P needs to be on both circles.

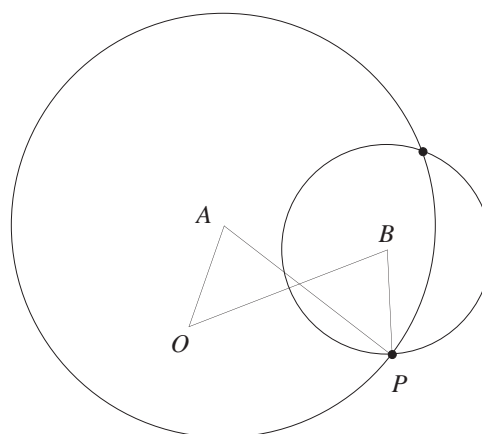
That narrows it down to two points (because there are only two points that are on both circles). Now, one of those points is the fourth vertex of a parallelogram whose sides are \overline{OA} and \overline{OB} :



Good P

What does it make?

But the other *doesn't* make a parallelogram:



Bad P

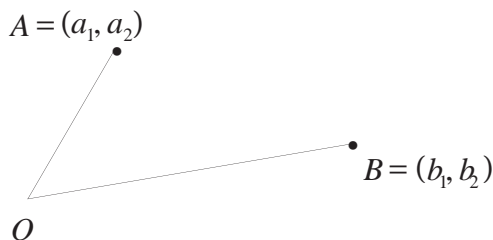
In the picture above, $PA = OB$ and $PB = OA$ without $OBPA$ being a parallelogram. In fact, $OBPA$ isn't even a legal *quadrilateral*, because it crosses itself, bowtie style. So, while a quadrilateral is a parallelogram if the opposite sides are congruent, a bowtie can have the same property (opposite sides congruent) without being a parallelogram. What distinguishes the bowtie from the parallelogram is that it crosses itself somewhere other than at a vertex.

This might seem like a small, technical worry. Given any points A and B , a simple picture will show that $A + B$ is the vertex of the parallelogram and not the vertex of the bowtie. But proofs that make use of the special properties of diagrams can get one into trouble. If you told a computer to locate a point P so that $PA = OB$ and $PB = OA$, it would not, without a lot of extra built-in smarts, be able to distinguish the parallelogram P from the bowtie P .

If the computer knows only what you learned in this module so far (midpoint and distance formulas, something about equations, and how to add and scale points), it would not be able to distinguish the two P s. Of course, *you* know a lot more than that.

It is possible to make things precise and provide an airtight proof of Theorem 5.4. One way to do it uses algebra and coordinates. Here's a sketch of how it might go:

Suppose our two points are $A = (a_1, a_2)$ and $B = (b_1, b_2)$.



In the coordinate game, one often has to solve for coordinates, rather than points. Choosing the notation to use to make them generic coordinates can make life a little easier or a little harder.

We're looking to find all points P so that $PA = OB$ and $PB = OA$. Let $P = (x, y)$, and let's set up some equations. Because $PA = OB$,

$$\sqrt{(x - a_1)^2 + (y - a_2)^2} = \sqrt{b_1^2 + b_2^2}.$$

Square both sides to get

$$(x - a_1)^2 + (y - a_2)^2 = b_1^2 + b_2^2.$$

Because $PB = OA$,

$$\sqrt{(x - b_1)^2 + (y - b_2)^2} = \sqrt{a_1^2 + a_2^2}.$$

Square both sides to get

$$(x - b_1)^2 + (y - b_2)^2 = a_1^2 + a_2^2.$$

So, we have two equations that x and y must satisfy:

$$(x - a_1)^2 + (y - a_2)^2 = b_1^2 + b_2^2$$

$$(x - b_1)^2 + (y - b_2)^2 = a_1^2 + a_2^2.$$

There are only two unknowns in these equations; a_1 , a_2 , b_1 , and b_2 are all constants.

1. Look at these equations for a minute. Without doing a lot of work, show that $x = a_1 + b_1$, and $y = a_2 + b_2$ is a solution. So, one solution to the original problem is

$$P = (a_1 + b_1, a_2 + b_2) = A + B.$$

2. Suppose $A = (3, 4)$ and $B = (12, 5)$. Write out the equations on the previous page in this case and show that the coordinates of $A + B$ make the equations work. Can you find other values that work?

According to Problem 1, one solution to our system

$$(x - a_1)^2 + (y - a_2)^2 = b_1^2 + b_2^2$$

$$(x - b_1)^2 + (y - b_2)^2 = a_1^2 + a_2^2$$

is $P = (a_1 + b_1, a_2 + b_2) = A + B$. It turns out that the system has another solution. Finding it requires some patience.

3. **Write and Reflect** How would you go about solving this system?

$$(x - a_1)^2 + (y - a_2)^2 = b_1^2 + b_2^2$$

$$(x - b_1)^2 + (y - b_2)^2 = a_1^2 + a_2^2$$

Take some time to play with the system. Expand the left sides, for example, and then add or subtract the equations to eliminate some of the messiness. And show, for example, that the two equations can be combined to produce

$$x(a_1 - b_1) + y(a_2 - b_2) = (a_1^2 + a_2^2) - (b_1^2 + b_2^2).$$

One way to find the other solution is to use a computer algebra system.

Well, it turns out that actually *finding* the other solution with what you currently know isn't easy. But with patience and a lot of algebraic manipulation, the other solution turns out to be

$$x = \frac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{(a_1 - b_1)^2 + (a_2 - b_2)^2} (a_1 - b_1)$$

$$y = \frac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{(a_1 - b_1)^2 + (a_2 - b_2)^2} (a_2 - b_2).$$

And this can be written as a single point P :

$$P = \left(\frac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{(a_1 - b_1)^2 + (a_2 - b_2)^2} \right) (A - B).$$

The stuff in front of $A - B$ is just a number—a complicated number, but just a number.

That is, one solution to our system is $A + B$, and the other turns out to be a very complicated multiple of $A - B$. Summarizing, we have another theorem:

THEOREM 5.10

If A and B are points, there are two points P so that $PA = OB$ and $PB = OA$. They are $A + B$ and $k(A - B)$, where

$$k = \frac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

4. Suppose $A = (3, 4)$ and $B = (2, 5)$. Calculate the value of k in Theorem 5.10 in this case, and show that $P = A + B$ and $P = k(A - B)$ are the two points that have the property that $PA = OB$ and $PB = OA$. Which one makes a parallelogram, and which one makes a bowtie?

5. Look at the form of the constant k :

$$k = \frac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

Is k ever 0? When? What does this say about the parallelogram? About the bowtie?

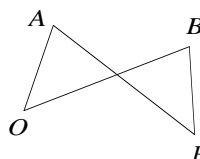
Does the bowtie become a necktie? Modeling this in geometry software gives you a feeling for what's going on.

6. What if you *knew* that the mysterious missing solution was of the form $k(A - B)$, for some number k . Could you find out what k would have to be? Hint: In Problem 3, you came up with the equation

$$x(a_1 - b_1) + y(a_2 - b_2) = (a_1^2 + a_2^2) - (b_1^2 + b_2^2).$$

Replace x with $k(a_1 - b_1)$ and y with $k(a_2 - b_2)$ and solve for k in terms of a_1 , a_2 , b_1 , and b_2 .

7. Suppose P is the bowtie-producing vertex; that is, suppose $PA = OB$ and $PB = OA$.



Copy the picture on the previous page and draw in \overline{OP} and \overline{AB} . Show that $OPBA$ is an isosceles trapezoid.

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WAYS TO THINK ABOUT IT

Sometimes, mathematical insights develop in a tangled web, mixing known facts, suspected facts, and things you *wish* were true. For example, faced with the problem of finding all the points P so that $PA = OB$ and $PB = OA$, you set up two equations in two unknowns:

$$\begin{aligned}(x - a_1)^2 + (y - a_2)^2 &= b_1^2 + b_2^2 \\ (x - b_1)^2 + (y - b_2)^2 &= a_1^2 + a_2^2.\end{aligned}$$

You know one solution (namely $(a_1 + b_1, a_2 + b_2)$), and you know there is another solution. If you subtract the equations, you get (as in Problem 6)

$$x(a_1 - b_1) + y(a_2 - b_2) = (a_1^2 + a_2^2) - (b_1^2 + b_2^2).$$

Problem 7 tells you that if P is the bowtie-producing vertex, then $OPBA$ is an isosceles trapezoid.

Here's where the tangled web comes in. Supposing that Theorem 5.4 is true (it said that O , A , $A + B$, and B form a parallelogram), Problem 4 in Investigation 5.16 implies that, since $\overrightarrow{AB} \parallel \overrightarrow{OP}$, there is a number k so that

$$k(A - B) = P - O = P.$$

Thus,

$$P = k(a_1 - b_1, a_2 - b_2) = [k(a_1 - b_1), k(a_2 - b_2)].$$

Replace x by $k(a_1 - b_1)$ and y by $k(a_2 - b_2)$ in

$$x(a_1 - b_1) + y(a_2 - b_2) = (a_1^2 + a_2^2) - (b_1^2 + b_2^2)$$

and solve for k . This gives you the formula in Theorem 5.10. So, assuming what you wanted to prove and working backwards, you are led to the insight ($P = k(A - B)$) that leads to "the right answer."

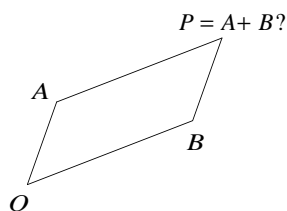
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It's precisely this kind of convoluted reasoning that is so hard to model on a computer.

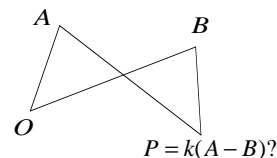
8. Does the previous discussion lead to a proof of Theorem 5.10? If so, write it up as one. If not, explain why and explain what else would have to be done to make it a proof.

Was there ever any doubt?

So, Theorem 5.10 gives you two points that satisfy $PA = OB$ and $PB = OA$. Which one makes a parallelogram and which one makes a bowtie? Of course the suspicion is that $P = A + B$ makes the parallelogram and $P = k(A - B)$ makes the bowtie.



$PA = OB$ and $PB = OA$



$PA = OB$ and $PB = OA$

Look at the two pictures. If you could show that the $k(A - B)$ solution gave a figure that crossed itself, then that value of P would be the bowtie, so the other value of P (namely $A + B$) would be the parallelogram. Well, a bit more algebra does the trick. Rather than go through all the details, here is the result without proof.

This means that $P = k(A - B)$ produces a figure that crosses itself.

THEOREM 5.11

Using the notation of Theorem 5.10, let $P = k(A - B)$. If s is the number

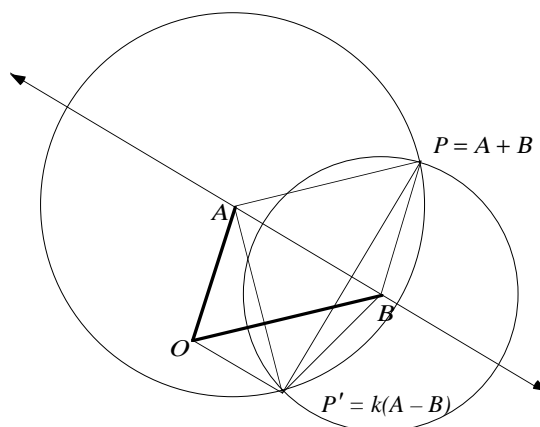
$$s = \frac{k}{k-1},$$

then \overline{AP} and \overline{OB} intersect at sB .

If you don't know how to show that a point is on \overline{AP} , look at Problem 3 in Investigation 5.15 for a hint.

9. If $A = (3, 4)$ and $B = (2, 5)$, calculate the values of k and s in Theorem 5.11. If $P = k(A - B)$, show that sB is on both \overline{OB} and \overline{AP} .

10. Challenge Here's the situation:



Show that \overleftrightarrow{AB} is the perpendicular bisector of $\overline{PP'}$. This implies that if you reflect P' over \overleftrightarrow{AB} , you get P . Using the fact that $OP'BA$ is an isosceles trapezoid, show that $OBPA$ is a parallelogram.

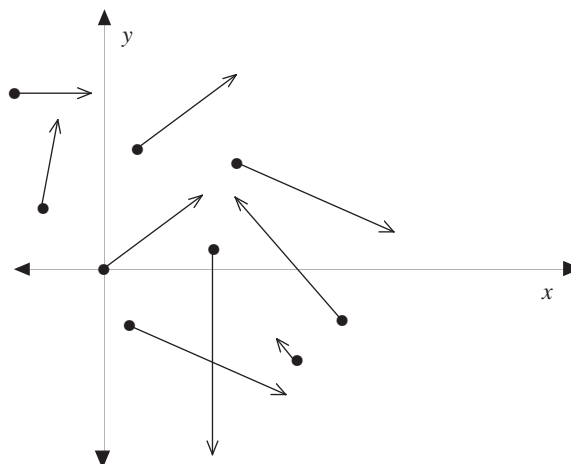
11. Write and Reflect This was a very long excursion into some pretty technical details. Summarize what has gone on in your own words. What exactly was the glitch in the original proof of Theorem 5.4? How did the glitch get resolved? What other mathematics did you learn in the course of resolving the glitch?

GETTING STARTED WITH VECTORS

In the last few investigations, you studied the geometry behind adding and scaling points. There's a very convenient language for using what you've learned. It involves the notion of *vector*.

In physics, people distinguish a “speed” (like 30 mph) from a “velocity.” A velocity has a direction as well as a size, so that 30 mph northeast is different from 30 mph due south. This idea can be modeled on a coordinate system by thinking about line segments that have a *direction*:

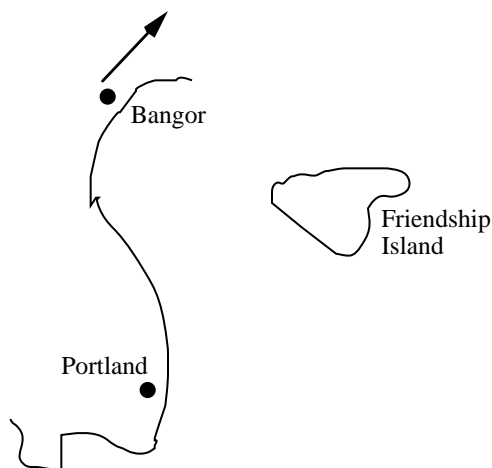
mph = miles per hour



Some directed segments

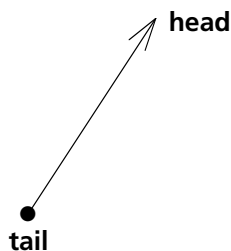
Each arrow in this figure can stand for a velocity. The length of the segment is the speed, and the direction of the segment tells you which way the velocity is “headed.” The arrows are often called *vectors*.

For navigators, vectors might stand for trips or currents or wind velocities. A wind velocity of 10 mph at a “bearing” of 40° from north could be represented like this:



1. The *length* of the arrow represents the speed of the wind, and the *direction* of the arrow shows the direction of the wind.
 - a. On a copy of the picture, draw a vector that represents a 20 mph wind with the same direction as the one in the picture.
 - b. On a copy of the picture, draw a vector that represents a 40 mph wind with the opposite direction from the one in the picture.

When speaking of winds, navigators might be most interested in the magnitude of the wind (10 mph, for example) and the direction of the wind (40° from north, for example). It makes much less sense to talk about the “place from where the wind starts.” In other words, the tail of the vector really doesn’t matter. A 10 mph wind at 40° from north is the same whether it blows while you are sitting in Bangor or Portland or on a pier on Friendship Island. So the vectors in this picture are somehow the same.



In geometry, it turns out to be useful *not* to call them the same. They’re not “equal,” because they’re not absolutely identical. For us, vectors will have a definite location. A vector requires an *initial point* (a “tail”) and a *terminal point* (a “head”). Instead

Sometimes, the kind of vectors used in this module are called “located vectors.” The navigator’s vectors (for example, wind) are sometimes called “free vectors.”

For the rest of this module, forget about rays; \overrightarrow{AB} will stand for the vector with tail A and head B .

$\overrightarrow{B(A+B)}$ means the vector whose tail is B and whose head is $A+B$.

of saying that certain vectors in the picture on page 131 are the same, you could say that they are “equivalent.” So, if A and B are points, the “vector from A to B ” can be thought of as the arrow that starts at A and ends at B . The notation for this vector is \overrightarrow{AB} . Notice that this is also the notation for the ray that starts at A and goes through B . Here’s some practice that will help you get used to the idea:

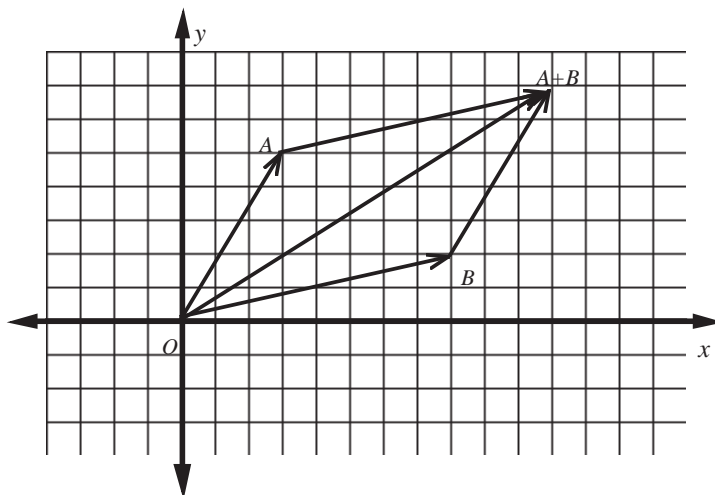
2. Let $A = (5, 1)$, $B = (-2, 5)$, $C = (9, -2)$, and $D = (16, -6)$. On a coordinate system, draw each of the following vectors:

- \overrightarrow{AB}
- \overrightarrow{CD}
- \overrightarrow{AC}
- \overrightarrow{BA}
- \overrightarrow{OA}
- \overrightarrow{OB}
- $\overrightarrow{B(A+B)}$
- $\overrightarrow{A(A+B)}$
- $\overrightarrow{O(B-A)}$

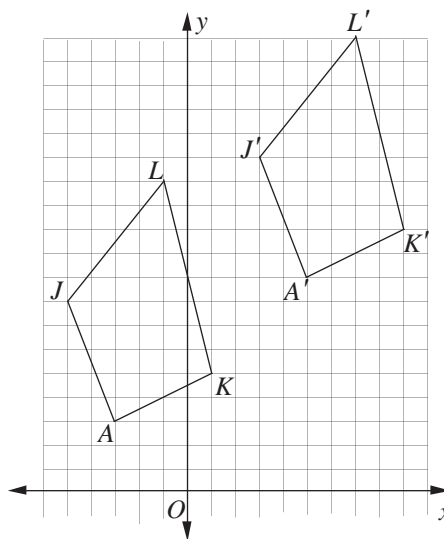
- Which of the vectors in Problem 2 have the same direction? Which have opposite directions?
- If $A = (5, 3)$ and $B = (8, 7)$, find a vector that starts at the origin and has the same length and direction as \overrightarrow{AB} .
- If $A = (5, 3)$ and $B = (8, 7)$, find a vector that starts at the origin, has the same direction as \overrightarrow{AB} , and is twice as long as \overrightarrow{AB} .
- Suppose you are working on a coordinate system. If $A = (3, 5)$ and $B = (8, 1)$, find two points C and D (neither at the origin) so that the vector from A to B is equivalent to the vector from C to D .
- Here’s a fancy pair of sentences:

$\overrightarrow{A(A+B)}$ is equivalent to \overrightarrow{OB} , and $\overrightarrow{O(A+B)}$ is one diagonal of the parallelogram whose vertices are O , A , $A+B$, and B . In addition, \overrightarrow{OA} is equivalent to $\overrightarrow{B(A+B)}$.

This is all pictured in the figure below. Copy the picture and label the five vectors that are mentioned in the sentences on the previous page.



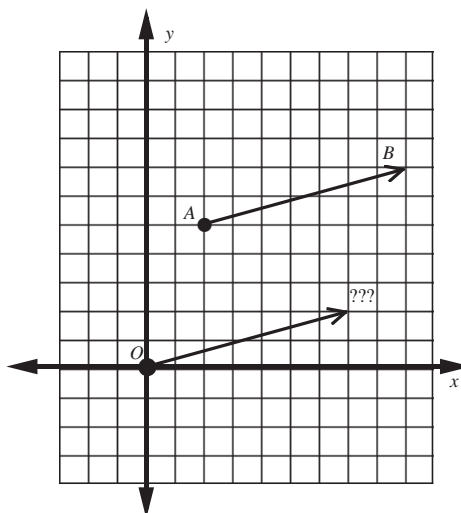
8. If $R = (7, 2)$ and $S = (15, 6)$, find the head of a vector that starts at $(4, -3)$ and is equivalent to the vector from R to S .
9. Copy the picture below and draw some equivalent vectors that suggest the movement of one polygon onto the other. Using coordinates, how do you know that all your vectors are equivalent?



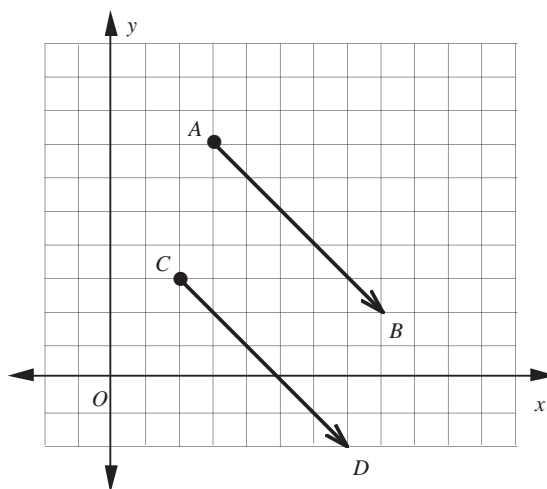
Remember, this means that,
for example, $T((4, 2)) =$
 $(4, 2) + (10, 5) = (14, 7)$.

This picture will give you a
feel for the function T
defined by the rule
 $T(X) = X + (8, 5)$.

10. True or false? If A and B are any points, the vector from O to B is equivalent to the vector from A to $A + B$. Explain.
11. Suppose that T is the function that moves points according to the rule $T(X) = X + (10, 5)$. Pick two points A and B . Draw the vectors from A to $T(A)$ and from B to $T(B)$. Show that these vectors are equivalent.
12. Plot a dozen or so random points. Translate each of them by $(8, 5)$. Draw a vector from each of your points to the corresponding translated points. Are all of the vectors you drew equivalent? Explain your answer.
13. If A and B are any two points, there is a vector that starts at O and is equivalent to the vector from A to B . Find a way to calculate the coordinates of the head of the vector with tail O below.



14. Find a way to tell if the two vectors below are equivalent just by looking at their coordinates (and doing some calculations with them).



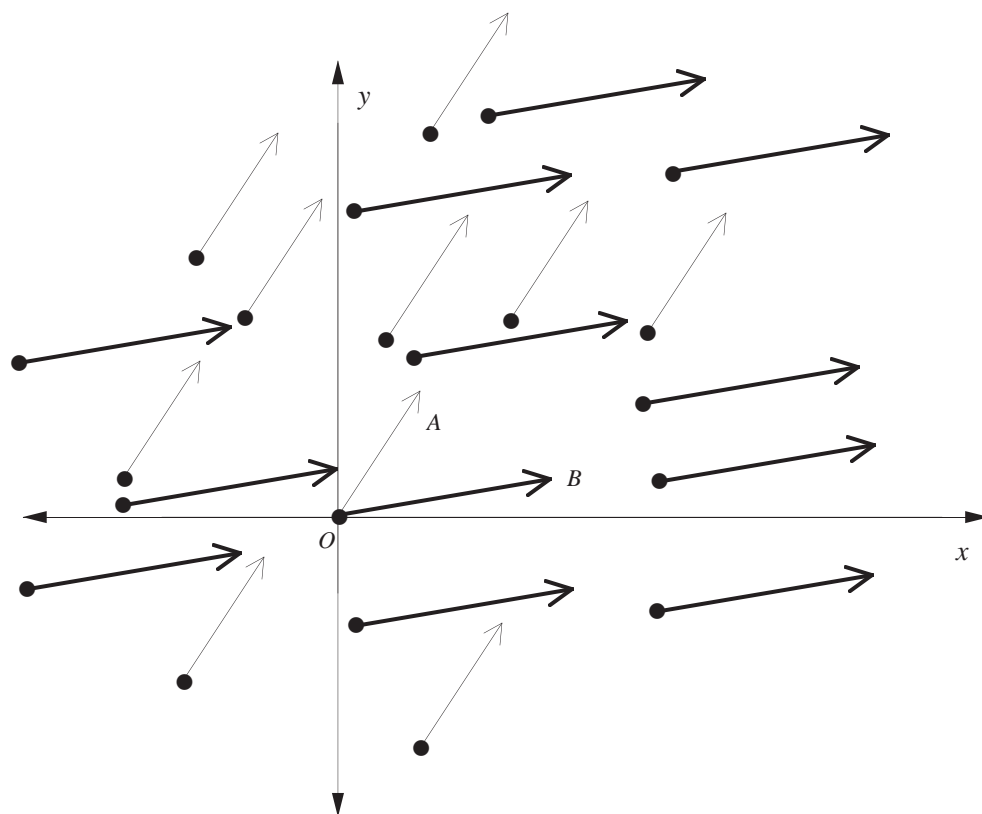
15. Prove the following theorem:

THEOREM 5.12

If A and B are points, the vector from O to $B - A$ is equivalent to the vector from A to B .

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WAYS TO THINK ABOUT IT

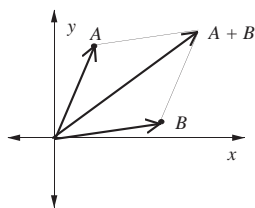
Because of Theorem 5.12, many people think of vectors starting at the origin as “representatives” for whole classes of vectors, all equivalent.



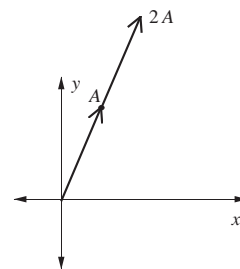
Two families of vectors represented by \vec{OA} and \vec{OB}

It sometimes brings clarity to a problem if you draw vectors from the origin to all the important points. For example, we discovered the

parallelogram law for adding by drawing vectors from the origin to A , B , and $A + B$. We learned how to scale a point A by drawing a vector from the origin through A .

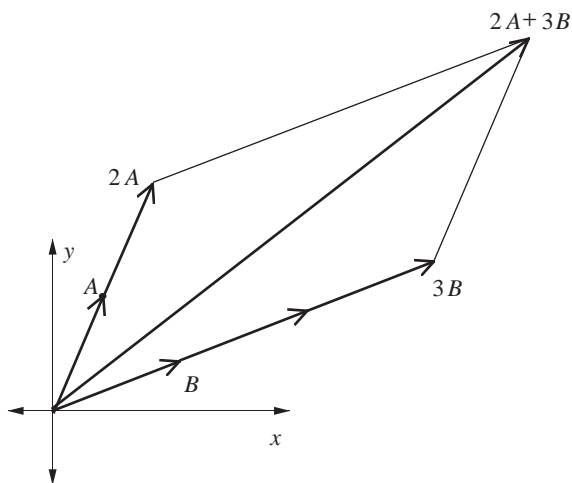


Adding



Scaling

With these two properties in place, forming various combinations of points can be pictured as stretching and adding “vectors”:



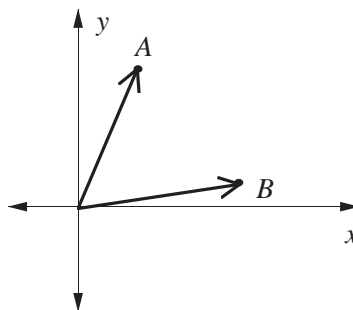
The next problem set asks you to try visualizing the algebra of points as “vector algebra.”

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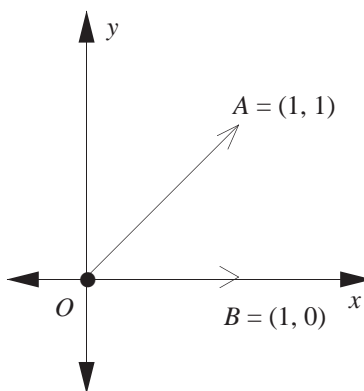
If you get stuck, try Problem 16 with actual coordinates. Even if you don't get stuck, draw a picture.

Think of $2A + 3B$ as “two A s and three B s.” Go out two A s from the origin, go out three B s from the origin, and draw the diagonal of the parallelogram.

16. If A and B are points, the vector from the origin to $A + B$ is one diagonal of the parallelogram whose sides are \overrightarrow{OA} and \overrightarrow{OB} . Show that the other diagonal is (as a vector) equivalent to the vector from O to $B - A$.
17. In the picture below, use “vector algebra” to locate
- a. $2A + 3B$;
 - b. $3A - B$;
 - c. $-2A - \frac{1}{2}B$;
 - d. $0.7A + 3.6B$.



18. In the picture below, the head of vector \vec{OA} is $(1, 1)$ and the head of vector \vec{OB} is $(1, 0)$.



Find x and y so that $x\vec{OA} + y\vec{OB}$ equals

- a. $(2, 1)$;
- b. $(4, 3)$;
- c. $(11, 1)$;
- d. $(5, 6)$;
- e. $(0, 8)$;
- f. $(1, -10)$.

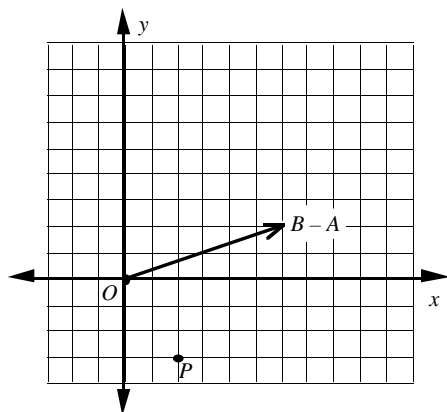
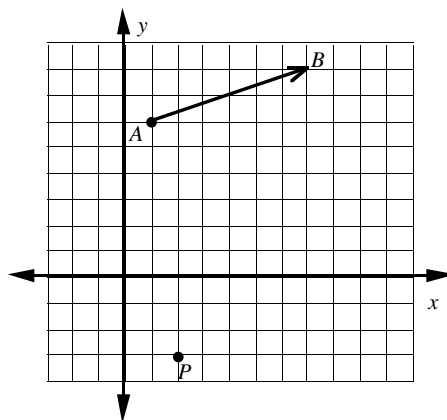
Suggestion: Draw the two vectors and the points on a piece of graph paper. Then think about how you would arrive at each of the points if you could only travel in the directions allowed by the two vectors.

HEAD MINUS TAIL

Theorem 5.12 says that \vec{AB} is equivalent to the vector from O to $B - A$. Some people think of this as “moving \vec{AB} to the origin” They say, “To move \vec{AB} to the origin, I just have to make the vector from O to $B - A$.”

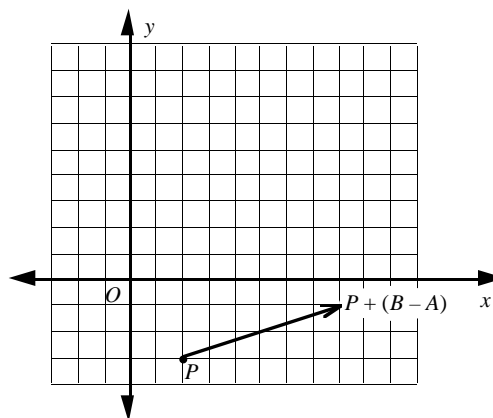
Other people say it this way: “To move a vector to the origin, just subtract the tail from the head.” The next problems show you how this technique can be applied to the geometry of vectors.

- 19. a.** To move a vector around, first move the vector to the origin, and then move it to the new spot. In the picture below, you can move \overrightarrow{AB} to the origin simply by calculating “head minus tail.” What does that give you?



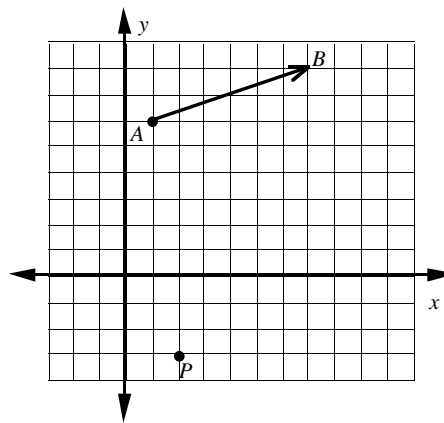
Move the vector to the origin ("head minus tail") and then move it back out by translating.

- b. Now just take the vector anchored at the origin and move it to P by adding P both to the head and to the tail. But adding P to the tail (which is just O) produces P . And adding P to the head produces the head of the vector you want. What is it?

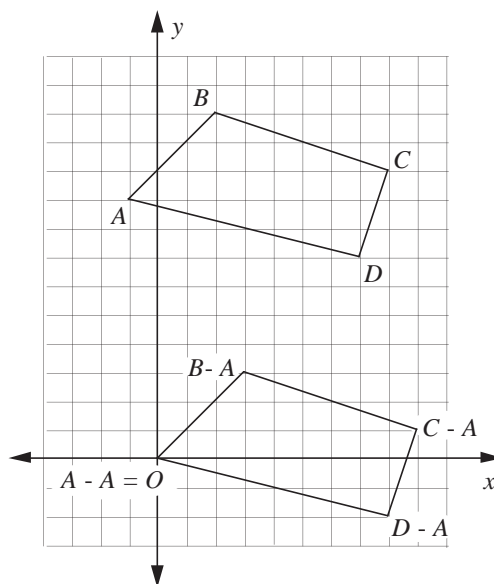


20. Find (and draw) a vector equivalent to \overrightarrow{AB} that

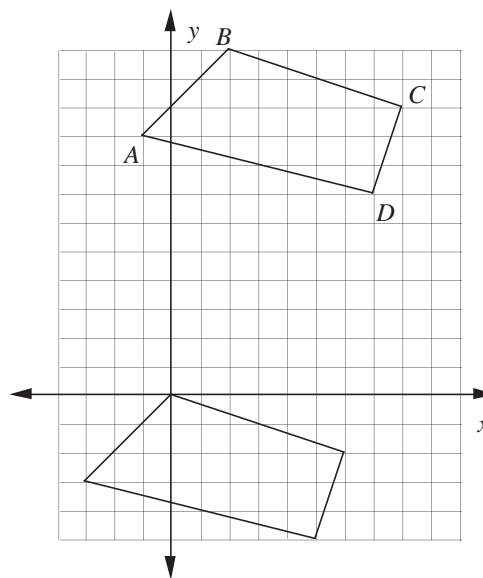
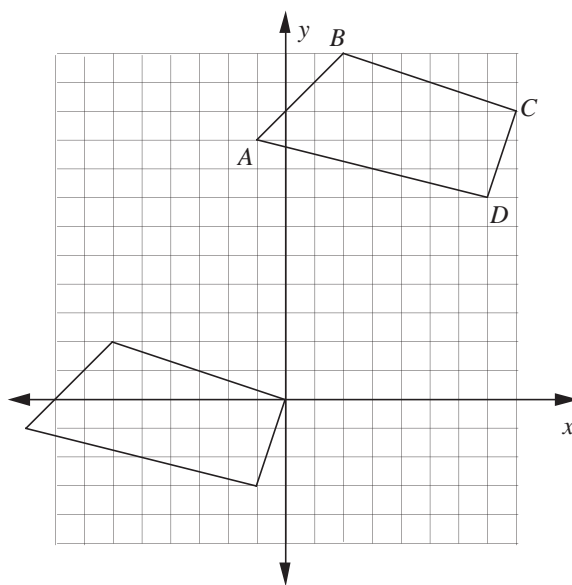
- a. Starts at $C = (8, 3)$;
- b. Starts at $J = (-8, 3)$;
- c. Starts at $K = (0, 3)$;
- d. Ends at O ;
- e. Ends at $C = (8, 3)$;
- f. Starts at B .

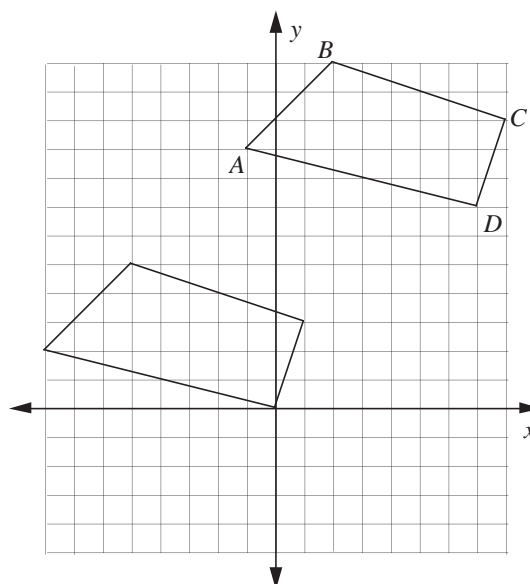


21. Suppose $A = (3, 4)$, $B = (9, 0)$, $C = (-1, 2)$, and $D = (5, -2)$. Show that \vec{AB} is equivalent to \vec{CD} by moving both vectors to the origin. Show also that \vec{AC} is equivalent to \vec{BD} .
22. The “move it to the origin” technique is useful in situations that don’t even seem to involve vectors. For example, a quadrilateral can be moved to the origin by subtracting any one of its vertices from all the rest. There are, then, four ways to do it. In each of the figures below and on the pages that follow, find the coordinates of the quadrilaterals at the origin. You *could* do this by counting squares (except some of the figures run off the graph paper), but it’s easier to do the arithmetic with the vertices of $ABCD$.



A to O

*B to O**C to O*



D to O

Since it's so easy to do, many people use the move-it-to-the-origin technique as a way to test vectors for equivalence rather than calculating heading and length. The strategy is this: Suppose you have two vectors and you want to see if they are equivalent. Move each of them to the origin (by finding “head minus tail” for each of them), and see if you get the same thing.

Head minus tail test Two vectors are equivalent if and only if head minus tail on one vector gives you the same result as head minus tail on the other. In symbols: \overrightarrow{AB} is equivalent to \overrightarrow{CD} if and only if

$$B - A = D - C.$$

23. Prove the following theorem:

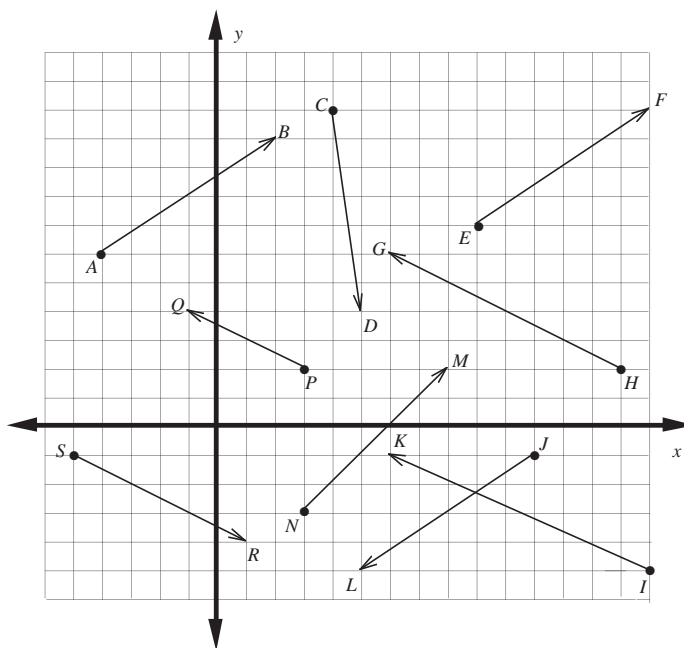
If you get stuck, try it with actual coordinates. Even if you don't get stuck, draw a picture.

THEOREM 5.13

Adding the same point to the tail and the head of a vector produces an equivalent vector.

Calculating head minus tail (which is the same as moving a vector to the origin) can be used to compare vectors for other things besides equivalence. The next problem asks you to extend the head minus tail test to cover more general relations.

24. Here are some vectors:



- a. Which vectors are parallel?
- b. Which are parallel in same direction?
- c. Which are parallel in opposite directions?
- d. Which are equivalent?
- e. Come up with “head minus tail” tests for two vectors to be
 - parallel in same direction;
 - parallel in opposite directions.

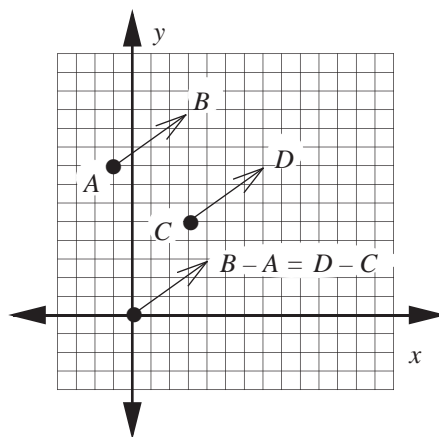
SUMMARY

This investigation has been concerned with the interplay between the geometry of vectors and the arithmetic of points. Here's a summary:

Equivalent Vectors

English: \overrightarrow{AB} is equivalent to \overrightarrow{CD} (same magnitude and direction).

Diagram:

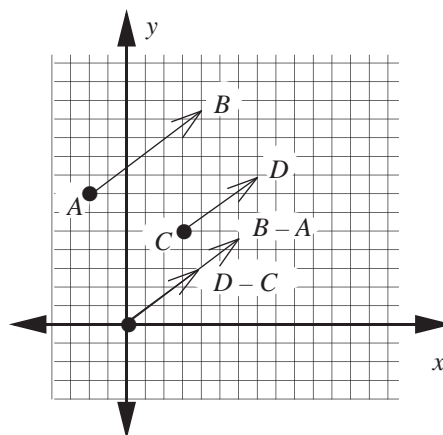


Arithmetic with Points: $B - A = D - C$

Parallel in the Same Direction

English: \overrightarrow{AB} is parallel to \overrightarrow{CD} in the same direction.

Diagram:

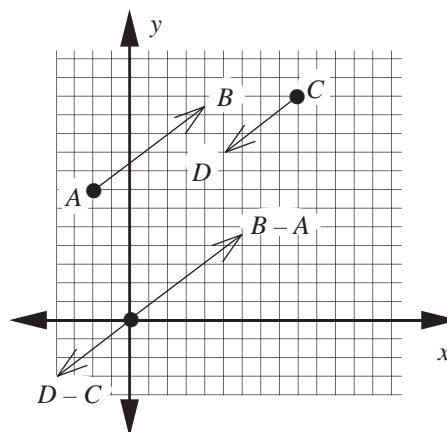


Arithmetic with Points: $B - A = k(D - C)$ for some positive number k .

Parallel in Opposite Directions

English: \overrightarrow{AB} is parallel to \overrightarrow{CD} in the opposite direction.

Diagram:



Arithmetic with Points: $B - A = k(D - C)$ for some negative number k .

All this can be summarized in one “megatheorem”:

THEOREM 5.14

Two vectors \overrightarrow{AB} and \overrightarrow{CD} are parallel if and only if there is a number k so that

$$B - A = k(D - C).$$

If $k > 0$ the vectors have the same direction. If $k < 0$ the vectors have opposite directions. If $k = 1$, the vectors are equivalent.

CHECKPOINT.....

“Not equivalent” is not a sufficient answer to this problem.

25. If $A = (3, 5)$, $B = (7, -2)$, $C = (9, 1)$, and $D = (5, 8)$, is the vector from A to B equivalent to the vector from C to D ? If so, why? If not, what *would* you call them?

26. If $A = (3, 5)$, $B = (7, -2)$, $C = (9, 1)$, and $D = (17, -13)$, describe the relationship between the vector from A to B and the vector from C to D .

27. If A , B , C , and D are the points in Problem 26, T is the function defined by the formula

$$T(X) = 2(X - A) + C.$$

What happens if you apply T to the vector from A to B ?

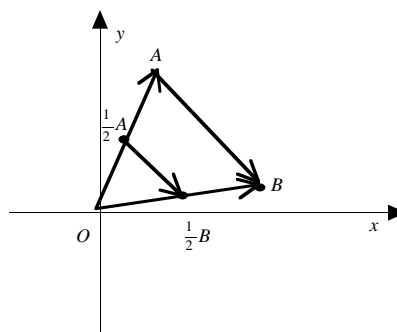
28. Find and label four points A , B , C , and D so that the vector from A to B is equivalent to the vector from C to D . Without labeling any other points, find another pair of equivalent vectors. Explain why what you say is true.

You now have a stash of theorems that allow you to use vectors to talk about coordinates, equivalence, parallelism, midpoints, distance, and directions. It's amazing how vector methods can simplify proofs of what are otherwise complicated theorems. For example, do you remember the Midline Theorem?

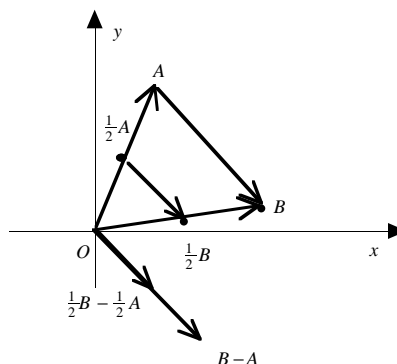
The line segment that connects the midpoints of two sides of a triangle is parallel to the third side and half as long.

A vector proof might go like this:

Take two sides of the triangle to be \vec{OA} and \vec{OB} . Then the midpoints are labeled in the picture:



Now, translate \vec{AB} and the midline to the origin:



We have

$$\frac{1}{2}B - \frac{1}{2}A = \frac{1}{2}(B - A),$$

and, by Theorem 5.14, $\frac{1}{2}(B - A)$ is parallel to $B - A$ and half as long.

Use any method you like,
but vectors and the algebra
of points come in very
handy.

This last section closes the module with a collection of problems that allow you to apply what you know. Use any method you like.

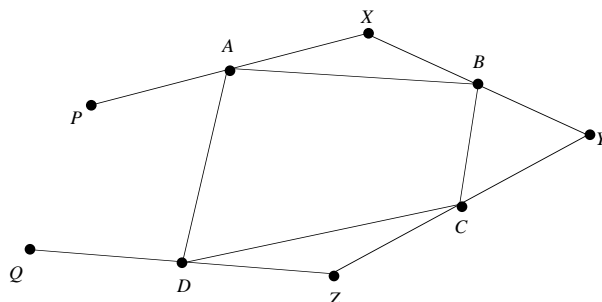
1. Show that the midpoint of \overline{AB} is $\frac{1}{2}(A + B)$.
2. Show that the point $\frac{2}{3}A + \frac{1}{3}B$ is on \overline{AB} , $\frac{1}{3}$ of the way from A to B .

Hint:

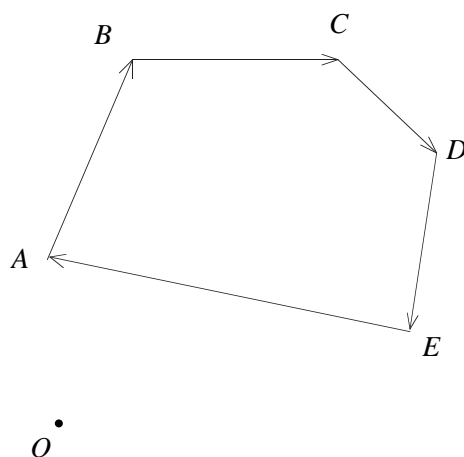
$$\frac{2}{3}A + \frac{1}{3}B = A + \frac{1}{3}(B - A).$$

3. What point is on \overline{AB} , $\frac{1}{4}$ of the way from A to B ?
4. Suppose $A = (3, 5)$ and $B = (7, 1)$. Find C if B is the midpoint of \overline{AC} .
5. Suppose $A = (0, 8)$, $B = (3, 12)$, $C = (12, -1)$, and $D = (6, -5)$. Show that the quadrilateral formed by joining the midpoints of $ABCD$ is a parallelogram.
6. Suppose A , B , C , and D are arbitrary points that are the vertices of a quadrilateral.
 - a. Express the midpoints of the sides of the quadrilateral in terms of A , B , C , and D .
 - b. Show that the quadrilateral formed by joining the midpoints of $ABCD$ is a parallelogram.
7. A British friend of the authors posed the following problem (sort of a converse to Problem 6):

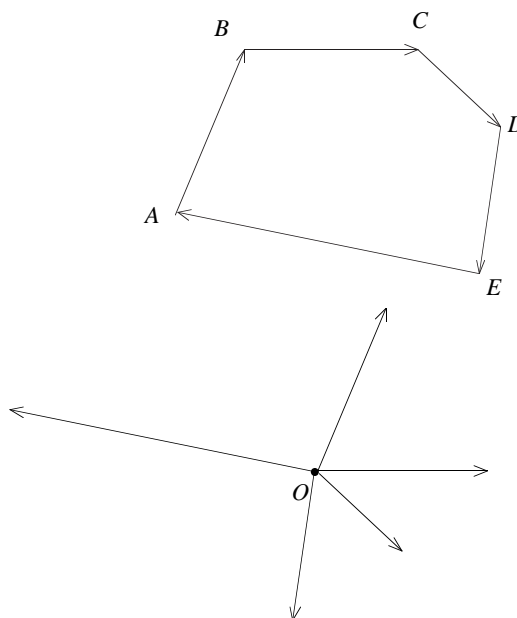
Draw an arbitrary quadrilateral $ABCD$ and place a point, labeled P as in the figure on the next page, anywhere you like. Construct a segment through A with P as an endpoint and A as its midpoint. Now begin at the endpoint of this new segment and construct another segment, this time with B as the midpoint. Continue doing this until you construct a segment with D as the midpoint. The finishing point of this whole journey around $ABCD$ is labeled Q in the figure.



- a. Draw vector \overrightarrow{QP} . If you change P 's location, \overrightarrow{QP} changes location but does not change direction or length. That is, all possible vectors \overrightarrow{QP} are equivalent. Prove this. (Hint: Show that $P - Q$ can be written solely in terms of A , B , C , and D .)
 - b. How can you make \overrightarrow{QP} have length 0? That is, when does $P = Q$?
8. Here's a picture of a polygon whose sides are vectors:



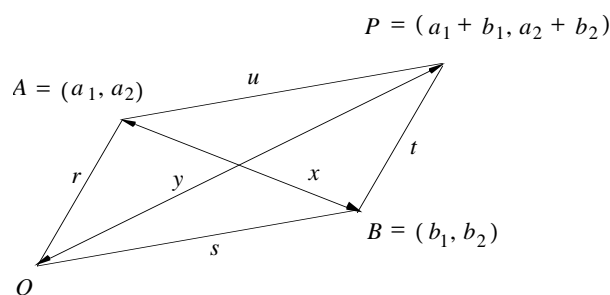
Move each side of the polygon (each vector) to the origin, to get five vectors, each with tail O .



- a. Label the heads of each of these vectors.
 - b. Show that the sum of these vectors is O .
9. Use vectors to show that the diagonals of a parallelogram bisect each other.

Hint: You can assume that one vertex of the parallelogram is O . If A and B are the vertices adjacent to O , the fourth vertex is $A + B$.

10. Show that in a parallelogram, the sum of the squares of the lengths of the diagonals is the same as the sum of the squares of the lengths of the sides.



$$x^2 + y^2 = r^2 + s^2 + t^2 + u^2 ?$$

If you get stuck, try it with actual coordinates. Even if you don't get stuck, draw a picture.

11. Suppose A and B are points. Using vectors, explain how to locate each of these points:
- $\frac{1}{3}A + \frac{2}{3}B$
 - $\frac{2}{3}A + \frac{1}{3}B$
 - $\frac{1}{4}A + \frac{3}{4}B$
 - $\frac{3}{4}A + \frac{1}{4}B$
 - $\frac{3}{5}A + \frac{2}{5}B$
 - $kA + (1 - k)B$ (here, $0 \leq k \leq 1$)
12. Let A , B , and C be points, and let $P = \frac{1}{3}(A + B + C)$. Show that in $\triangle ABC$, P is $\frac{2}{3}$ of the way from any vertex to the midpoint of the opposite side.
13. George has a way to find the population center for three cities, each of the same size. He puts the cities on a coordinate system, adds the coordinates of the three cities, and scales by $\frac{1}{3}$. In what sense is George's point the "population center?"

Hint:

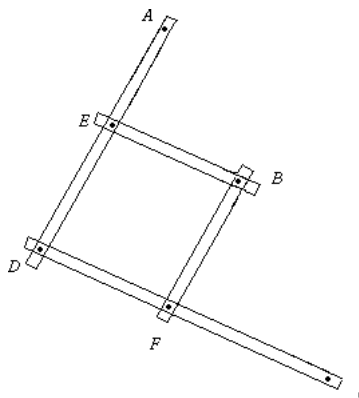
$$\frac{A+B+C}{3} = \frac{1}{3}B + \frac{2}{3}\left(\frac{A+C}{2}\right)$$

In solving this problem, you are finding the *center of gravity* of the triangle. This is the spot where you could balance the triangle on a pencil or on your fingertip.

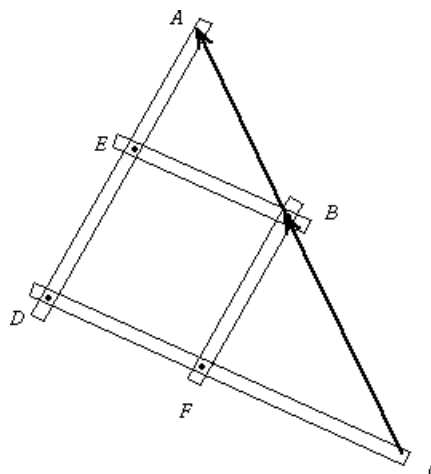
Try it with some cities near you.

Points E and F are midpoints of \overline{DA} and \overline{DO} .

14. Martha extends George's method by allowing for cities of different sizes. She first coordinatizes the map, scales the coordinates of each city by its population, adds the results, and then divides the resulting point by the sum of the populations of all three cities. In what sense is Martha's point the "population center?"
15. Why does a pantograph work? That is, suppose you build a collection of linkages like the one below. The rods are pivoted at D , E , B , and F . Why, if you nail O to a table and trace out a figure with point A , does point B trace out a smaller but similar figure?



Hint: Does the picture below suggest any explanations to you?



***MODULE OVERVIEW AND
PLANNING GUIDES***

ABOUT THE MODULE	T₂
MAIN TIMELINE PLANNING CHART	T₃
ALTERNATE TIMELINES	T₅
ASSESSMENT PLANNING	T₇

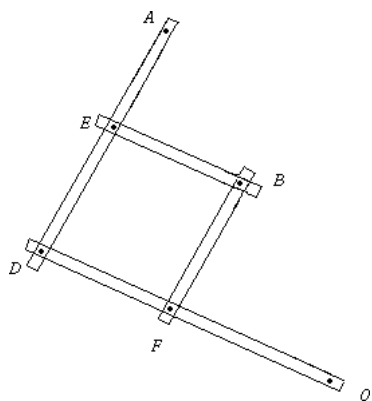
ABOUT THE MODULE

This module provides an introduction to analytic geometry—from the notion of “what is a coordinate system?” through standard Cartesian coordinates in two and three dimensions, and continuing on to the introduction of vectors. Habits of mind include picturing the visible and the invisible, proving, and using different systems.

The first section of the module, “Introduction to Coordinates” (Investigations 5.1–5.6) provides an introduction to coordinates in one, two, and three dimensions. This section is ideal for a high-school class not yet familiar with coordinates (in an algebra, geometry, or integrated mathematics course). It is also suitable for a middle-school class. Topics include plotting points and lines, deriving the formulas for midpoint and distance, working with points in 3-dimensional space, thinking about collinearity, and numerical properties of the coordinates of basic shapes.

The second section, “Coordinates, Geometry, Points, and Vectors” (Investigations 5.7–5.12) explores how algebraic manipulations of coordinates transform points and objects in a Cartesian coordinate system. High-school classes already familiar with coordinates might start here, skipping Investigations 5.1–5.6. (Investigation 5.7 includes a review of topics in the first section of the module.) Topics include translations, scaling, and an introduction to vectors.

The third section, “Coordinates, Algebra, and Vectors” (Investigations 5.13–5.19) formalizes the notion of vectors and the operations of adding and scaling points, develops the proofs of two theorems about the algebra of points, and ends with students using vectors to prove some standard Euclidean geometric results and to analyze how a pantograph works. It is an ideal exploration for more advanced classes, for classes working seriously with the notion of proof, and for teacher preparation classes.



a pantograph

MAIN TIMELINE PLANNING CHART

Working through the complete module, from the earliest introduction to coordinates through the much more advanced coordinate and vector proofs in the section “Coordinates, Algebra, and Vectors” would take an average class 10 or 11 weeks.

Investigation	Description	Key Content	Pacing
5.1 Location in Space	This investigation gets students thinking and visualizing locations. The investigation introduces vocabulary: <i>coordinates</i> , <i>frames of reference</i> , and <i>origin</i> .	<ul style="list-style-type: none"> • visualizing • location of object • some vocabulary 	1 day
5.2 Locating Points in Two Dimensions	Students learn the basics of Cartesian coordinates and plotting points. Polar coordinates provide another example of a two-dimensional coordinate system.	<ul style="list-style-type: none"> • coordinate sense • plotting points • horizontal and vertical lines • midpoint and distance • circles 	2 days
5.3 Lines, Midpoints, and Distance	This problem-based investigation contains a brief review of plotting points, and then moves on to develop what we call “coordinate sense.”	<ul style="list-style-type: none"> • horizontal and vertical lines • collinearity • midpoints and distance 	5 days
5.4 Formulas for Midpoint and Distance	Students develop formulas for midpoint and distance in the plane. They also explore the equivalence of two different methods for finding midpoints.	<ul style="list-style-type: none"> • formulas for midpoint and distance 	5 days
5.5 Coordinates in Three Dimensions	Students extend their “coordinate sense” to three dimensions. They build models of \mathbb{R}^3 and think about coordinate descriptions for shapes in space.	<ul style="list-style-type: none"> • models of 3-dimensional Cartesian coordinates • 3-dimensional coordinates 	3 days
5.6 Shapes in the Plane and in Space	Students further develop their coordinate sense by looking at shapes in terms of coordinates in two and three dimensions. They examine “rules” applied to coordinates that produce lines, circles, and other shapes. Students develop their own tests for collinearity of three points.	<ul style="list-style-type: none"> • coordinate sense • coordinate description of shapes • collinearity 	4 days

Investigation	Description	Key Content	Pacing
5.7 Introduction to Coordinates and Vectors	This investigation is a brief review of coordinates—plotting points, horizontal and vertical lines, midpoints, distance, and circles—for students who have not covered Investigations 5.1–5.6.	<ul style="list-style-type: none"> • coordinate sense • plotting points • horizontal and vertical lines 	2 days
5.8 Stretching and Shrinking Things	Students learn two methods for scaling objects in the plane: operating on the coordinates of points on the figures and stretching and shrinking vectors.	<ul style="list-style-type: none"> • scaling with coordinates • scaling with vectors 	1 day
5.9 Changing the Location of Things	Shapes are translated in the coordinate plane by arithmetic operations on the points that make them up. Students learn a new notation to describe arithmetic operations on points (both translations and dilations): $(x, y) \mapsto (ax + b, cy + d)$.	<ul style="list-style-type: none"> • translating using coordinates • new notation: $(x, y) \mapsto (ax + b, cy + d)$ 	2 days
5.10 Pictures from Rules, Rules from Pictures	This investigation brings together ideas about translations, dilations, arithmetic operations on the coordinates of points on figures, and the notation introduced in the previous two investigations. It is the perfect opportunity for assessment.	<ul style="list-style-type: none"> • translations and dilations • arithmetic operations on coordinates of points on figures • assessment opportunity 	2 days
5.11 Scaling Points	This investigation introduces scaling figures both by arithmetic operations on the coordinates and by using vectors. Self-similar figures are informally introduced.	<ul style="list-style-type: none"> • scaling with coordinates • scaling with vectors • collinearity of all multiples of a point 	4 days
5.12 Adding Points	This investigation also introduces the “parallelogram rule” for adding vectors, the idea (presented very informally) of two vectors spanning the plane, and the process reversing a translation by adding the negative of the scalars used to translate.	<ul style="list-style-type: none"> • translating by adding points • translating with vectors • congruence of translated figures 	3 days
5.13 Making Things Precise	Students prove theorems about coordinates in this investigation, first by showing that they are true for specific points and then by using generic points to prove the theorems. Some proofs are given, and students give reasons for each step. Students also prove things from scratch.	<ul style="list-style-type: none"> • proof 	3 days

Investigation	Description	Key Content	Pacing
5.14 Using the Theorems	This investigation relies on two theorems presented in the previous investigation. Students use these theorems as well as a new theorem to prove several properties of adding and scaling points.	<ul style="list-style-type: none"> • proof • parallel lines 	2 days
5.15 The Algebra of Points	This investigation asks students to prove some properties of points as well as a general case of the Midline Theorem.	<ul style="list-style-type: none"> • proof • Midline Theorem 	2 days
5.16 More on Scaling Points	This investigation generalizes a theorem from the previous investigation. Students use their knowledge of square roots and absolute value to make two generalizations of the theorem.	<ul style="list-style-type: none"> • proof 	2 days
5.17 More on Adding Points	Students are asked to fix a glitch in the proof of an earlier theorem, namely that there are two points for which $PA = OB$ and $PB = OA$, but only one of them is the point $A + B$.	<ul style="list-style-type: none"> • proof 	2 days
5.18 Vectors and Geometry	This investigation introduces students to the language and use of vectors. Students are asked to discover and apply several properties of vectors.	<ul style="list-style-type: none"> • formalization vectors • properties of vectors 	4 days
5.19 Using Vectors to Solve Problems	This investigation includes a collection of problems that require several skills including adding and scaling points, vector algebra, and several theorems pertaining to coordinates and vectors that have been studied in this module.	<ul style="list-style-type: none"> • proof • Midline Theorem • parallelograms • centers of triangles • pantagraphs • assessment opportunity 	2–5 days

ALTERNATE TIMELINES

Quick Trip

- 5.1–5.6 (12 days)
- 5.7–5.12 (10 days)
- 5.13–5.19 (18 days)

Quick Trip through Most of the Module (8 weeks)

This route is suitable for a class that works at a slightly faster rate, where a substantial number of problems is assigned for homework. In Investigation 5.1–5.6, students work many problems for homework. Investigation 5.11 is shortened by skipping a few problems (Problems 4, 6, or 7) and doing fewer examples in Problems 1, 2, and 5.

For Investigation 5.12, follow this outline: Day 1 Problems 1a, 1b, 1d, 1f, 1g, 2 (don't do all the examples), 4, 6, 8, skip 9 or use for a discussion, and 10; Day 2 Problems 11–14, 18, and 20–22 as reading or homework. Day 3 is for a short assessment, using Problems 5, 7, and 16. Investigations 5.13–5.19 should be completed as described in the Main Timeline Planning Chart; much of the material is too subtle to rush through.

Knows Coordinates

- 5.1–5.7 (5 days)
(optional)
- 5.8–5.12 (12 days)
- 5.13–5.19 (18 days)

Pick and Choose

- 5.1–5.6 (8 days)
- 5.8–5.12 (8 days)
- 5.13–5.19 (18 days)

The Class Already Knows Coordinates (6–7 weeks)

A class that already knows the basics of coordinates—plotting points, finding mid-points, and the distance formula—can skip most of the section “Introduction to Coordinates.” If you want to review some of these topics but not cover them in depth, you can pull selected problems from Investigations 5.1–5.6 and create a 3–5 day lesson using them and Investigation 5.7. Investigations 5.8–5.19 can be completed as described in the Main Timeline Planning Chart.

Pick and Choose (7–8 weeks)

This timeline can be used if your class knows a little about the coordinate plane and may not be able to work as quickly as the “Quick Trip” timeline suggests. Skip Investigation 5.1 and the first 5 problems of Investigation 5.2. Students can complete 8 or 10 of the remaining problems in class and for homework. On the second day, students can work on Investigation 5.3, Problems 3–9, 16–19, 22–29 in class and for homework. You can use the “Checkpoint” (Problems 31–40) *later* as assessment. Investigation 5.4 may take 2 days. Skip to the subscript problems and complete a few of those. Students complete Problems 12–15, 17–21, 23–25, 27, 30, 31, 33, and 35. Problems 32 and 35 can be used as assessment. You can spend one day on an assessment of Investigation 5.3 and 5.4 and one day on Investigation 5.5, completing Problems 3–7, 11, 12, and 16–19. Problems 20 and 21 can be saved for assessment. Investigation 5.6 will take 2 class days. Students complete Problems 1, 2, 4–7, 9, 11, 13, 14, 17, and 30–22. Use Problems 13, 18, 19, 23 and 24 as assessment.

Investigation 5.8 can be completed in one day. Complete all of Investigation 5.9. You may choose to skip Problem 10 in Investigation 5.10 and use Problems 11 and 12 as assessment. Investigation 5.11 can be covered in 2 days. Students complete Problems 1–3, 5–8, (you can choose to skip either 6 or 7), 11, 12, 14–16. Problems 9, 10, 17, 18, 20, 22 and 23 can be used as assessment. Three days may be allotted for Investigation 5.12 (to allow for spillover from Investigation 5.10 or 5.11). In Problems 1 and 2 you may choose only a few of the examples. Students complete Problems 3–5, 7, and 9–12. You may use Problems 14–16 as assessment. Complete Problems 18–20, read “Ways to Think About It” and do problems 21 and 22.

Complete Investigations 5.13–5.18 as described in the Main Timeline Planning Chart.

Coordinates Only

- 5.1–5.6 (20 days)
- 5.7–5.19 (8 days)

Coordinates Only (8 weeks)

You may use this timeline if you choose not to study vectors at all. Investigations 5.1–5.6 are to be completed as described in the Main Timeline Planning Chart. Investigation 5.7 is optional. In Investigation 5.8, leave out the “vector method for stretching and shrinking.” In Investigation 5.11, skip “Another Way to Do It” which includes Problem 5. Skip the “vectors” and the “parallelogram rule” methods mentioned in “Ways to Think About It” in Investigation 5.12. Skip Investigations 5.13–5.19 altogether.

ASSESSMENT PLANNING

Throughout the entire module, we recommend that students keep a notebook of:

- daily homework and other written assignments
- a list of vocabulary, definitions, and theorems that emerge during classwork and homework

What to Assess

- The student can use the Cartesian coordinate system in one, two, and three dimensions.
- The student can plot points given coordinates and can read coordinates from points on a coordinate grid.
- The student can find the midpoint between two points and the distance between two points. The student can relate the midpoint formula to similar triangles, and distance to the Pythagorean Theorem.
- The student has developed some “coordinate sense”: He or she can tell what is the same about all the points on a horizontal or vertical line or in a given region.
- The student can scale figures by operating on the coordinates of points.
- The student can translate figures by operating on coordinates.
- The student can interpret and write rules using “ \mapsto ”, e.g., $(x, y) \mapsto (2x + 1, 2y)$.
- The student can describe which operations on coordinates produce congruent figures, which produce similar figures, and which produce distortions.

- The student can give examples of equivalent rules—rules that show a different process with the same result.
- The student can operate on points: scaling a point by a number, adding two points, adding a point to every point on a figure, etc.
- The student can explain that all multiples of a point lie along a line through that point and the origin and can approximately locate multiples of a point using that fact. The student can describe where cA is located, relative to A and O , for different values of c including: $c > 1$, $0 \leq c \leq 1$, $c < 0$, $c = 0$, $c = 1$.
- The student can describe and implement two ways to add points: adding the coordinates and using the “parallelogram method.”
- The student can tell if two vectors are parallel, and if two vectors are equivalent (parallel and the same length).
- The student can read, complete, and create proofs using coordinates, vectors, and the algebra of points.

QUIZZES AND JOURNAL ENTRIES

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
5.2	Write up an answer to one of Problems 1–5 from Investigation 5.1. Allow students to submit a rough draft with a day to revise and make the final draft.	See Teaching Notes.
5.3	Problems 10, 11, or 12: Write an answer with explanation <i>Write and Reflect</i> Problem 24, or Problems 28 and 29	
5.4	Problem 12 Ask students to rewrite in their own words the proof of the midpoint formula. Ask students to begin with the Pythagorean Theorem and turn it into the distance formula.	See Teaching Notes.

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
5.6	<i>Write and Reflect</i> Problems 12 and 13, or 18 and 19, or 23 and 24.	Have students do a careful write-up of solutions to the six <i>Write and Reflect</i> problems.
5.7	Problems 2, 4, 5, 6, 8, and 10	Test the students with problems in Investigation 5.7 which review all the important ideas from the first six investigations.
5.8	<p>Problem 4 (essential)</p> <p>Problems 6–8 (scaling using coordinates vs. scaling using vectors)</p> <p>Have students present solutions to Problems 5a and b, or to Problem 8</p>	
5.9	<p>Problem 4 (essential)</p> <p>Problem 8 (essential) Include conjectures about: What operations on coordinates produce congruent figures? Similar figures? Distortions?</p> <p>Have students present solutions to: Problem 1 (different students for each of a, b, c, and d), Problems 3a, b, and c, or Problems 5a and b.</p>	
5.10	<p>Problem 4 (matching rules to pictures)</p> <p>Problem 9 (translating rules to algebra)</p> <p><i>Write and Reflect</i> Problem 11</p>	

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
5.11	<p><i>Write and Reflect</i> Problems 3 and 4 Problems 14 and 15 (scaling by 1 as an identity) Problems 18 and 20 (essential) Problem 23 (essential)</p> <p>Have students present solutions to: Problems 12a and b, Problems 13a and b, Problem 17, Problem 24 (different students for each of a, b, c, d) or Problem 25.</p>	
5.12	<p>Problems 9, 11, and 12 (adding points to a figure, reversing a process, and adding 0 as an identity) Problems 14 and 16 Problem 22</p> <p>Have students present solutions to Problem 19a and b.</p>	<p>Ask students to state and explain the midpoint formula.</p> <p>As a test, ask students to apply the distance and midpoint formulas in problems like the three listed at the end of this chart.*</p>
5.13	<p><i>Write and Reflect</i> Problems 9 (proof) and 10 Problems 20 and 22 (proofs) <i>Write and Reflect</i> Problem 24</p> <p>Have students present: solutions to Problem 24; a proof of Theorem 3, using the write-up to Problem 9 as notes; a proof of Theorem 4, including restating “thing 1” proved in the Student Module and students’ proofs of “thing 2” from Problem 20.</p>	
5.15	<p>Problem 1 (proof) Problems 4 and 5</p> <p>Have students present solutions to Problem 5.</p>	

Investigation	Journal Suggestion or Presentation	Quiz Suggestion
5.16	Problems 2 and 3 (proofs) Have students present solutions to Problems 1, 2, and 3 (together or as separate presentations, depending on your students).	
5.17	Problem 3 Problem 8 (proof) Problem 11 Have students present: solutions to Problem 16; a proof of Theorem 10; or a proof of Theorem 11 (general case).	
5.18	Problem 9 Problem 13 Problem 17: As an added challenge, have students find x and y so that $xOA + yOB = (a, b)$. Problem 22 (proof)	
5.19		See the Teaching Notes for details on how to create a final assessment using the problems in the Student Module.

- *1. Draw a right triangle on a coordinate grid; you may choose the coordinates. Find the coordinates of the midpoint of the hypotenuse. Show that this point is the same distance from all three vertices. Would this be true for all quadrilaterals?
- *2. Draw a parallelogram on a coordinate grid; you may choose the coordinates. Find the coordinates of the midpoints of each side. Connect these midpoints to form a new quadrilateral. Show that this new quadrilateral is a parallelogram because opposite sides are congruent. Would this be true for all quadrilaterals?
- *3. Draw a parallelogram on a coordinate grid. Show that the diagonals of the parallelogram bisect each other.

LOCATION IN SPACE

OVERVIEW

This section of the module introduces the basics of plotting points and the Cartesian coordinate system in two and three dimensions. Midpoint and distance formulas are developed by students. They also work on developing what we call “coordinate sense,” fluency and facility with coordinates:

- What is the same about all the points on a vertical line? On a horizontal line?
- How can I tell if three points are on the same line?
- How can I find points on a circle?

Investigation 5.1 describes the general notion of a “coordinate system” and location of points within a system. It introduces several nonstandard, one-dimensional systems along with the standard number line to emphasize the abstract idea. You may want to start with this one- or two-day activity, or you might move directly to Investigation 5.2, which introduces two-dimensional systems including the standard Cartesian plane and polar coordinates.

This investigation gets students thinking about and visualizing the locations of points within a coordinate system. The first two problems may not seem related to coordinates, but they are related to how location defines objects. (Are congruent squares, sharing a side but lying in different planes, really *different* squares?) The investigation introduces the following vocabulary: *coordinates*, *frames of reference*, and *origin*.

TEACHING THE INVESTIGATION

This is best taught as a full-class activity. You can state, or students can read, the questions and associated text. The class can brainstorm answers to Problems 1–6. Answers should emphasize the notion of reference point, and points that are in (or not in) the space. Following these problems, you may want to use the “For Discussion” questions to informally assess students’ thinking about location in a coordinate system. Individually, students can write a short answer to Problem 7 and try to come up with a solution for Problem 8.

This investigation could also be done in small groups, bringing students together for a discussion following group work on these problems.

ASSESSMENT AND HOMEWORK IDEAS

A formal assessment would be inappropriate at this early stage. Short answers to Problem 7 will reveal if students understand the general concept of coordinate systems and their dimensions.

If class time is completely taken up with the discussion of Problems 1–6, Problems 7 and 8 could be done for homework. If your class finishes the whole investigation, you can assign a short reading and writing activity from the beginning of Investigation 5.2 for homework. You could also ask students to write up their summary of the “For Discussion.”

LOCATING POINTS IN
TWO DIMENSIONS

Materials: graph paper

OVERVIEW

Students learn the basics of Cartesian coordinates and plotting points. Polar coordinates provide another example of a two-dimensional coordinate system.

TEACHING THE INVESTIGATION

If your class skipped Investigation 5.1, you should also skip problems 1–5; start instead with the section “One of Two Common Systems.”

If you will be doing problems 1–5 with your students, you can ask them to pick one and answer it in writing for homework the night before starting the activity.

If students have read and completed one of the “thought experiments” (Problems 1–5) for homework, you may want to spend some time discussing their responses. Consider uniqueness of labels in their coordinate systems and also reference points. Then, introduce the class to the Cartesian coordinate system and allow students to spend some time working on Problems 6–14 either individually or in pairs. For homework, students should finish through Problem 18.

Begin the second day with the “For Discussion” about Problem 13; encourage some early conjectures about operations on coordinates (a major topic in the second section of this module). Follow with the discussion about changing sign; answers should relate to Problem 18 from the homework. Depending on your class, you may want to spend the rest of the time working on the “Take It Further” problems and talking about polar coordinates, reading the “Perspective” essay about Descartes, or conducting a short assessment.

If students have not had experience with geometry software, the wording of the problem, “how a point moves,” may be confusing. Ask them instead, “how the location of a point changes.”

Note that these statements don’t depend on whether x and y are positive or negative.

Relate the discussion on changing signs back to Problem 10. If you start with two positive coordinates, what happens when you make the first one negative instead? What happens when you make the second one negative? What about both of them? You may want to introduce the following notation and definition of reflection:

- If $A = (x, y)$, then $B = (-x, y)$ is the reflection of A over the y -axis.
- If $A = (x, y)$, then $C = (x, -y)$ is the reflection of A over the x -axis.
- If $A = (x, y)$, then $D = (-x, -y)$ is the reflection of A over first one axis and then the other. (Some teachers refer to this as reflecting A over the origin.)

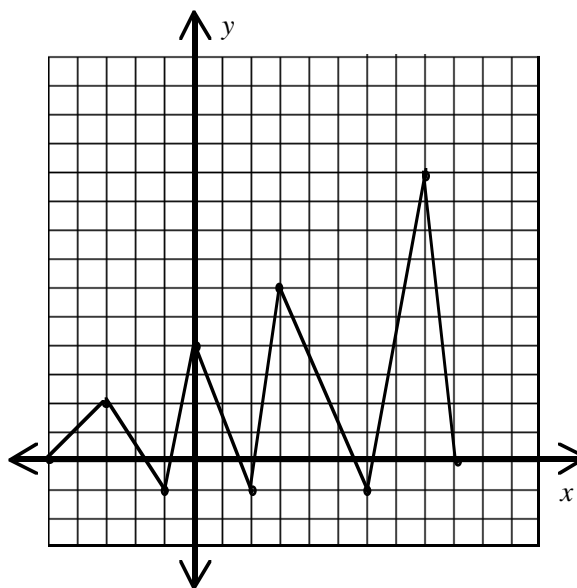
Problem 18 is another opportunity to revisit these ideas.

ASSESSMENT AND HOMEWORK IDEAS

Any of the problems are suitable for homework, especially Problems 11–18. If you want to conduct a short assessment on coordinate systems and plotting points, you might create a quiz with questions like these:

- Why is the location of points important?
- Describe at least one coordinate system you have learned about.
- Using Cartesian coordinates, label the points in the drawing below.

You can create any pictures you want for a problem like this, but be sure to label the points with letters so students can refer to them precisely in their answers.



- Describe the characteristics of x - and y -coordinates in each of the four quadrants in a Cartesian coordinate system.

SUPPLEMENTARY ACTIVITIES

To start developing an informal notion of function on the plane, you might try the following short activity:

For example, stand them at $(-5, 0)$, $(-4, 0)$, \dots , $(0, 0)$, \dots , $(4, 0)$, $(5, 0)$, and so on. Have students identify their starting and ending coordinates.

This will quickly get too big to be in the classroom. It's fun to do outside if you can, or to estimate how far away the person at $(5, 0)$ would be. Are you still on the school grounds?

Create a coordinate grid for your classroom. Have students stand along the x -axis of your classroom grid. Then ask them to follow and act out rules like the following:

- Make your second coordinate the same as your first.
- Make your second coordinate the negative of your first.
- Make your second coordinate twice as much as your first (or half as much, or any other multiple).
- Make your second coordinate the square of your first.
- For the person at the origin, make your second coordinate 1. For everyone to her right, make your second coordinate twice as much as the person before you. Where should the people to the left of the person at $(0, 1)$ stand? Will anyone have a second coordinate of zero? A negative second coordinate?

Afterwards, ask students to sketch (freehand) the various shapes in which they were standing.

USING TECHNOLOGY

Pieces of paper, the floor or wall of a room, or a planar space in your mind's eye can be divided up into the four quadrants of a Cartesian plane. Then the two-dimensional drawing or image of any object can be examined on that plane. Logo does the same thing with the screen of a computer, and you can take advantage of it if you program in Logo. The “home” position of the turtle is at the origin, with coordinates $(0, 0)$. Picture a horizontal line running across your screen and a vertical line running down the screen, each passing through “home.” Those are your coordinate axes.

1. Write a program in Logo to draw the x - and y -axes on your screen.

Logo also has `pos`, `xcor`, and `ycor` commands that return the value of the current position, the current x-coordinate or the current y-coordinate. You can use them to save your current position and return to it later.

2. You can use Logo's `setpos`, `setx`, and `sety` commands to tell the turtle exactly where to go on the screen.
 - a. Try the following commands. For each command, try it starting from "home" and from other locations on the screen.
 - `setpos [-20, 50]`
 - `setx 100`
 - `sety -60`
 - b. Try a few more examples of `setpos`, `setx`, and `sety`.
 - c. Explain what each command does to the turtle (both its location on the screen and the direction it points).
3. You can use `setpos`, `setx`, and `sety` to draw shapes exactly where you want them on the screen. Draw a quadrilateral with vertices at (90, 50), (-80, 30), (-120, -60), and (50, 10).
4. Use `setpos`, `setx` and `sety` to draw a square with sidelength 95 and one vertex at (45, 10).
5. Write a procedure that will draw a square with a sidelength that you input and one vertex at a point that you input.

LINES, MIDPOINTS, AND DISTANCE

Materials: graph paper

What's coming up?
Investigation 5.4 will
formalize midpoint and
distance into formulas.

**If you post important
results and ideas in your
class, you may want to add
these to the list; they will
be used often in the rest of
the module.**

OVERVIEW

This problem-based investigation contains a brief review of plotting points, and then moves on to developing what we call “coordinate sense.” Students will learn about

- horizontal and vertical lines;
- collinearity;
- midpoints;
- distance.

Before starting this investigation, students should be familiar with the coordinate plane and with plotting points, either from Investigation 5.2 or previous coursework. The Pythagorean Theorem is used to find distances. The term “congruent” is used in some problems.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Day 1: Students can do Problems 1–6 for homework the night before or in class on the first day. In either case, you will want to hold a class discussion to summarize some important ideas:

- On a horizontal line, all points have the same y -coordinate.
- On a vertical line, all points have the same x -coordinate.

Students then work on Problems 7–13 in class and then finish them for homework.

Day 2: Students work on Problems 14–20 individually or in small groups. End with another summarizing discussion:

- How do you find the distance between two points on a vertical or horizontal line?
- How do you find the midpoint of a horizontal or vertical segment?
- How do you find the distance if the two points are not on a vertical or horizontal line?

You might mention to your class that all distances are measured with respect to some unit (like feet or meters).

Problem 20, in particular, can help summarize a class discussion.

If there is time, students work on Problems 21–23 and then finish them for homework.

Day 3: Discuss the homework, especially Problem 23. Why do you get a circle? Students should realize (or you should point out) that this is exactly the definition of a circle: the set of points a fixed distance from a center point. As a follow-up, ask them to name six points that are five units from the point $(1, 1)$. Can they find a rule to generate points on a circle? (One example is $(\sqrt{x}, \sqrt{25 - x^2})$ for $x \leq 25$.)

Students then work on Problems 24 and 25, conclude with a discussion about distance, and use the Pythagorean Theorem to find distance.

Day 4: Students work on Problems 26 and 27 individually or in small groups. Follow this with a class discussion on how different students or groups solved the problems, allowing students to describe their methods for finding the midpoint of a nonvertical, nonhorizontal segment. Students should then *individually* write up Problems 28 and 29. They can work on Problem 30 with a partner and revise their methods if necessary. Homework: Problems 31–37.

Day 5: Discuss and correct the homework in class. These problems summarize some of the important ideas from this investigation; these ideas will be used throughout the module. Students work in small groups on Problem 38. End with a whole-class discussion:

- Describe all the points that are equidistant from P and Q ; what shape do they make? (They form the perpendicular bisector of \overline{PQ} .)
- Is $(13, 25)$ closer to P or Q ? Is $(-2, -2)$ closer to P or Q ?

Students then work on Problems 39 and 40 in small groups.

ASSESSMENT AND HOMEWORK IDEAS

Most of the problems in this investigation are suitable for homework, allowing for flexibility in how you move through the material. A few (Problems 24–25 and 28–30) require working with a partner at some point or discussion with others in the class, so these should be done in class.

The written instructions required for Problems 24 and 28 are ideal assessments, especially if students complete individual write-ups after small-group work on the problems. Checkpoint Problems 31–38 are a good final assessment for the investigation.

Alternatively,
Problems 31–37 could be a
short, in-class assessment
on day 5.

FORMULAS FOR MIDPOINT AND DISTANCE

Materials: graph paper

Side-Splitting Theorem: A line parallel to the base of a triangle cuts the other two sides proportionally. It is proved in the module *A Matter of Scale*.

See notes in the Solution Resource about Problems 23–26 for information about “weighted averages.”

OVERVIEW

Students develop formulas for midpoint and distance in the plane. They also explore the equivalence of two different methods for finding midpoints.

Investigation 5.3 should be completed before this one. (The methods for finding midpoint and distance that students found in Investigation 5.3 will be developed into formulas in this investigation.) Students should be familiar with the Pythagorean Theorem and the Side-Splitting Theorem.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Day 1: After students solve Problem 1 on their own (either in class or for homework the night before), ask two students to take the roles of Kesia and Paul and to read the dialogue out loud. Discuss the two methods for finding midpoints. Students then work on Problems 2 and 3.

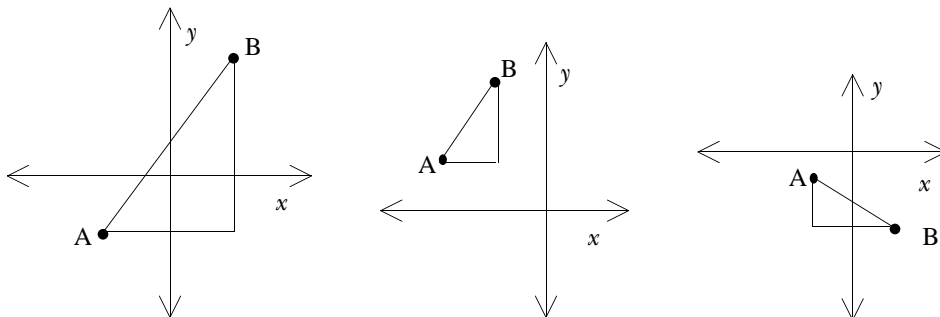
If students are unfamiliar with subscripts, read from that section aloud in class, answering questions and clarifying as necessary. Students then work on Problems 4–12 and finish them for homework.

Day 2: Discuss homework to make sure students are clear on the use of subscripts. Read through the proof of the midpoint formula and discuss it as a whole class. Students work on Problems 13–22, finishing them for homework.

Day 3: Discuss the homework. If your students have developed a reliable method for finding midpoints, move on to the “Take It Further” problems. Students can work individually or in groups to solve the problems. As a whole class, discuss the notion of “weighted averages” and why they work. Otherwise, move on to the distance formula.

Students should compare the formula given in the book to their own methods from Problem 24 in Investigation 5.3.

Day 4: Begin class with a discussion of the distance formula and how to derive it from the diagram in the Student Module. As part of the discussion, show students different arrangements of two points such as those shown below. Ask the class, “Given the formula in the Student Module, does it matter which is x_1 and which is x_2 ?”

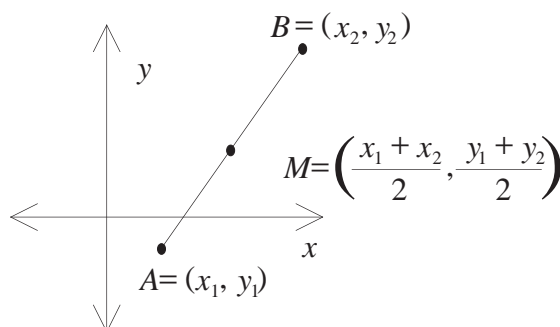


Students work on Problems 27–30 and finish for homework.

Day 5: Check homework as a class. Students then work in small groups on Problems 31–35. At the end of the day or the next day, groups present one of their solutions to the class. Homework: write up Problems 31–35 to turn in. Optional: choose either Problem 34 or 35 to prove.

Slope is introduced in Investigation 5.6. Some students will be familiar with this concept from their study of algebra.

Notes: A more traditional proof of the midpoint formula than the one provided in the Student Module uses the distance formula and slope rather than the intermediate steps of finding two other midpoints. If your students already know about slope, you may want to present this proof after it’s clear they know the distance formula.



$$\text{slope of } \overline{AB} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{slope of } \overline{MB} = \frac{y_2 - \frac{y_1 + y_2}{2}}{x_2 - \frac{x_1 + x_2}{2}} = \frac{y_2 - y_1}{x_2 - x_1}$$

So M is on the line containing \overline{AB} .

$$\begin{aligned} AM &= \sqrt{\left(x_1 - \frac{x_1 + x_2}{2}\right)^2 + \left(y_1 - \frac{y_1 + y_2}{2}\right)^2} \\ &= \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ BM &= \sqrt{\left(x_2 - \frac{x_1 + x_2}{2}\right)^2 + \left(y_2 - \frac{y_1 + y_2}{2}\right)^2} \\ &= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \end{aligned}$$

Why is $(x_1 - x_2)^2$ the same as $(x_2 - x_1)^2$?

So $AM = BM$. Therefore, M is the midpoint of \overline{AB} .

Problem 34 asks students to prove half of the Midline Theorem:

THEOREM *The Midline Theorem*

If students know about slope, it's not hard to prove the other half of the theorem as well.

A segment connecting the midpoints of two sides of a triangle is parallel to the third side and half as long.

In the *Connected Geometry* module *The Cutting Edge*, students prove this theorem using congruent triangles. If students have seen both proofs, it might be interesting to compare them: Which proof was easier to understand? Does one proof give you more insight into “why” than the other?

ASSESSMENT AND HOMEWORK IDEAS

Most of the homework suggested in “Teaching the Investigation” involves students finishing or writing up problems they began in class. You could also pull problems from the next investigation about midpoints and distance in three dimensions and assign them as homework here.

Presentations and write-ups of Problems 31–35 make an excellent assessment.

Investigation
5.5

Student Pages 39–44

COORDINATES IN THREE
DIMENSIONS

Materials: For 3D models:

- pencils and tape
- clay and toothpicks
- stiff paper and scissors

The day before: Get materials ready for building the models.

OVERVIEW

Students extend their “coordinate sense” to three dimensions. They build models of \mathbb{R}^3 and think about shapes in space in terms of coordinate descriptions.

Students should have had some work with the Cartesian coordinate plane (Investigations 5.3 and 5.4 or previous coursework) before doing this investigation.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Day 1: One of the most useful things you can do to help students visualize three-dimensional coordinate systems is to start out with them building their own models of the 3D coordinate system. Three models are suggested in “Supplementary Activities” below. Work on Problems 1–5. End class with a discussion about Problems 1–5 and how to visualize the shapes. Homework: Write up Problems 1–5 carefully and complete Problems 6–12. (You can skip Problems 9 and 10 if you think they will be esoteric.)

Day 2: Discuss the homework. Set up the classroom coordinates and answer Problems 13–15 as a class. Then move on to Problems 16–19 (if you didn’t assign them as homework in the last investigation). End class with a discussion comparing the three-dimensional methods to the two-dimensional methods for midpoint and distance.

As homework or a short assessment: Problems 20–22. (You may want to assign Problem 22 as extra credit.)

Students should be referring to their models for help with the visualization problems. For the homework, they may benefit from a model to use at home. The three-pencils model is sturdy, portable, and easy to recreate at home.

For the model of planes using stiff paper, you may want to add a “grid” by gluing graph paper onto each side of the three sheets of stiff paper. This requires a little care in making sure the slits are made along axes, but it helps some students with picturing points in space.

SUPPLEMENTARY ACTIVITIES: MODELING 3D COORDINATES

The following mini-projects are suggestions for making one or more models of a three-dimensional coordinate system to get a better idea of what it looks like. Try at least one of the three; you may want to keep one nearby to refer back to later.

The thickness of the pencils prevents this from being a perfect model. The third pencil can be made perpendicular to the other two, but it must be taped off to one side of their intersection.

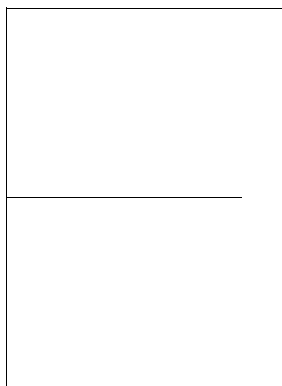
You'll need 27 balls of clay for a model with three balls to an edge; ask the class: "How many would you need for a model with 5 balls to an edge?"

The 3 axes: Place one pencil on top of another so that they cross near their middles and are perpendicular to each other. Connect them with tape, and then attach the third pencil to the intersection of the first two. It needs to be perpendicular to both of them. You now have a model of three-dimensional axes.

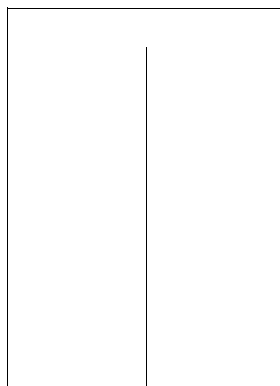
Lattice points: Begin with three or four clumps of different-colored modeling clay or gum-drops and a few boxes of toothpicks.

Roll small, marble-sized balls of clay, and connect them with toothpicks to build a cube. If you make this model with an odd number of balls to each edge you can place the origin in the center of the model. Of course, it doesn't matter which ball you designate as the origin, but it might be helpful to make it one color, use a second color for the three axes, and a third color to designate the points with nonzero coordinates.

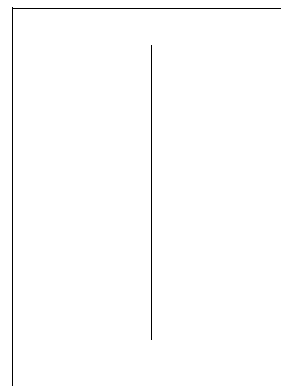
The 3 planes: Start with six sheets of graph paper, and three pieces of construction paper or other stiff paper. Glue a sheet of graph paper onto each side of each sheet of stiff paper. On the first sheet, cut it nearly (but not completely!) in half across the width. On the second sheet, cut it nearly in half lengthwise. On the third sheet, make a slit lengthwise along the middle. (See the pictures below.)



cut along the width



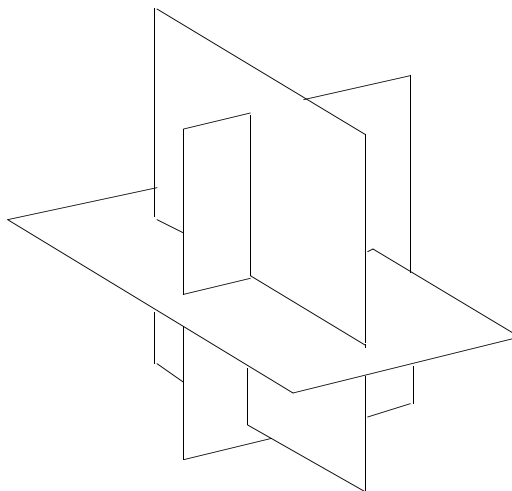
cut along the length



slit down the middle

In the first model, the pencils depicted the three axes. Here, the pieces of graph paper represent the intersecting coordinate planes in the three-dimensional coordinate system.

Now connect the three sheets together: The third sheet (with the slit) fits into the cut on the first sheet. The cut on the second sheet slides over the first sheet, and the ends slide into the slit on the third sheet. You may need to wiggle the papers around a bit to make them all perpendicular to each other. You now have a model of a 3D coordinate system.



ASSESSMENT AND HOMEWORK IDEAS

Homework and assessment are suggested in “Teaching the Investigation.”

**SHAPES IN THE PLANE
AND IN SPACE****OVERVIEW**

Students further develop their coordinate sense by looking at shapes in terms of coordinates in two and three dimensions. They examine “rules” applied to coordinates that produce lines, circles, and other shapes. Students develop their own tests for collinearity of three points.

Students should have familiarity with coordinates, some facility with algebraic expressions, and knowledge of the distance and midpoint formulas. Problems 11 and 13 ask about similarity and Problems 10 and 12 ask about congruence; your class can skip these problems if they haven’t studied the concepts yet.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Day 1: Students work on Problems 1–3 in small groups. At the end of the class, students can discuss circles and spheres in terms of coordinates: what is the same about all the coordinates of points on a circle centered at the origin? Centered at some other point? How does this extend to spheres? Homework: Problems 4–7. These are more difficult, but students should be able to come up with points that work by trial and error and at least to guess at the shapes.

Day 2: Discuss the homework—students should confirm that the results are ellipses for the first three problems (see the Solution Resource), and you may want to spend some time on Problem 7.

- If $x + y = 12$ and $x - y = 12$ are both lines, how do they differ? What about things that look like them (for example, $4x + 3y = 12$)? This is a good time to review the equation of a line if your students have seen it before, or to preview it if they have not.
- Why do the product and ratio give such different shapes? How does the line for $\frac{y}{x} = 12$ relate to (and differ from) the line $y - x = 12$?
- What shapes do you get with $xy = 12$? Can you explain what happens near the axes?

This is rich territory; you will have to decide how much of it to explore with your students and how much to leave without comment.

The triangle problems are ideal for assessing students' understanding of the use of the Distance Formula. You may want to alter the problems for use as a quiz.

Students work on Problem 8 alone and then trade pictures with partners. Students can then work on Problems 9–11 alone or with their partners. Write up Problems 12 and 13 for homework.

Day 3: Students work individually or in groups on Problems 14–19 about lines. End with a discussion of slope. The degree of formality here depends on your class. You may want to extend the idea to lines like $y = 2x + 3$ as well (especially if you discussed this formula for a line in the earlier class discussion). Write up Problems 18 and 19 for homework.

Day 4: Spend time checking students' methods for testing collinearity (Problem 19) and exploring why they do (or don't) work. Students with strategies that don't work can revise them for homework the next day. For the rest of the class, students work on Problems 25 and 26 (optional).

ASSESSMENT AND HOMEWORK IDEAS

Homework is suggested in “Teaching the Investigation” above. Almost any of the problems here are suitable for homework, allowing for flexibility in how you move through the material.

The six “Write and Reflect” problems (on congruence, similarity, and collinearity) are good assessments of students' ideas about coordinates and operations on them. These problems ask students to pull together ideas from the several problems that came before, so they are particularly useful as individual assessments if students worked on the previous problems in groups.

This investigation also provides another opportunity to assess students' use of some ideas from the previous investigation. In particular, several problems require use of the Distance Formula, but students must notice this on their own and choose to use it.

INTRODUCTION TO COORDINATES AND VECTORS

Materials: graph paper

OVERVIEW

In this section of the module, students learn about translations, dilations, and reflections of shapes by operating (arithmetically) on the coordinates of the points that make them up. New notation is introduced: $(x, y) \mapsto (2x + 3, 2y - 4)$ to describe transformations on points.

This section of the module includes a very informal introduction to vectors as “arrows” with length and direction, a notion which is developed further in the section, “Coordinates, Algebra, and Vectors.” Mathematicians will point out that vectors are not arrows: they are equivalence classes or translations. We believe students can see vectors that way. In the investigations in this module, students build up to those views a bit—first using vectors as translations, and later in the module thinking about equivalent vectors as those with the same length and direction, no matter where their “start point” is. But we assume this module is the first time your students have seen vectors, so our treatment remains informal. Students can focus on the big ideas, so that when they encounter vectors again—in calculus, physics, or linear algebra, for example—they are ready to think about them in the more formal ways.

This investigation is a brief review of coordinates—plotting points, horizontal and vertical lines, midpoints, distance, and circles—for students who have not been through Investigations 5.1–5.5 (or have not studied this material recently).

This investigation assumes familiarity with coordinates and plotting points. It focuses on a review of some big ideas and on coordinate sense.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Assign Problems 1–6 as homework the night before starting the investigation.

Day 1: Discuss Problems 1–6. Students then work on Problems 7–10 and hand in their written directions for finding midpoints and distances between two points.

Day 2: If there were any problems with directions for finding midpoint and distance, cover these two ideas more carefully (you may want to pull ideas or problems from Investigations 5.3 and 5.4). Otherwise, move on to Investigation 5.8.

Be sure to discuss the equations of vertical lines, horizontal lines, $y = x$ and $y = -x$, and the definition of circles as the set of all the points that are a fixed distance from a given point.

**STRETCHING AND
SHRINKING THINGS**

Materials: graph paper

OVERVIEW

Students learn two methods for scaling objects in the plane: operating on the coordinates of the points on the figures and stretching and shrinking vectors.

Familiarity with coordinates is required. Students should be able to complete all of the problems from Investigation 5.7 without trouble.

TEACHING THE INVESTIGATION

Problems 1–5 can be done for homework the night before you begin the investigation or in class. After working on these problems, students should discuss their rules for what happens to figures when the coordinates of their points are multiplied by some number (Problem 4). As a whole class, read about and discuss the “vector method” for scaling pictures. For the rest of class or for homework: Problems 6–8.

The big idea in all of these problems is that multiplying the x - and y -coordinates of each point on a figure by some positive number scales the figure by that number, so you end up with a similar figure. There are some subtleties to this idea, and it’s up to you how much you want to focus on these points:

- This is actually a dilation of the figure, not just a scaling of it, so it also moves the figure closer to the origin if the scale factor is less than 1 and farther from the origin if the scale factor is greater than 1.
- For some positive number a , if you multiply the x -coordinate by a and the y -coordinate by $-a$, you get a scaled and flipped (reflected over the x -axis) copy of the figure. If you multiply the x -coordinate by $-a$ and the y -coordinate by a , the figure is scaled and reflected over the y -axis. If both the x - and y -coordinates are multiplied by $-a$, the figure is scaled by a and reflected over both axes.
- If you multiply the x - and y -coordinates by different numbers, you get a distortion of the figure rather than a dilation; it is no longer similar to the original.
- Multiplying the x - and y -coordinates by 1 is the identity transformation; you get the same figure you started with.

What’s coming up? Vectors are used informally throughout the rest of this module. They are introduced here for the first time.

ASSESSMENT AND HOMEWORK IDEAS

Homework is suggested in “Teaching the Investigation” above. Ideas introduced here will be formalized (and assessed) later in this section of the module.

CHANGING THE
LOCATION OF THINGS

Materials: graph paper

OVERVIEW

Shapes are translated in the coordinate plane by arithmetic operations on the points that make them up. Students learn a new notation to describe arithmetic on points (both translations and dilations): $(x, y) \mapsto (ax + b, cy + d)$.

Students should have completed Investigation 5.7 and be comfortable with performing arithmetic operations on variables.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Day 1: Small groups work on Problems 1–5. Follow with a whole-class discussion on how to operate on coordinates to translate shapes in the plane.

Introduce such notation as $(x, y) \mapsto (ax + b, cy + d)$. Give a few examples like the following: How would you write “add 4 to the first coordinate and add 3 to the second coordinate?”

$$(x, y) \mapsto (x + 4, y + 3)$$

How would you write “multiply the first coordinate by 10 and the second coordinate by 75?”

$$(x, y) \mapsto (10x, 75y)$$

Then ask students to take five minutes to do a few examples on their own. You can use Problems 1–5 from this investigation and Problems 2a–c from Investigation 5.8 or make up your own. (All students need to do is translate the “multiply by” or “add to” statements into $(x, y) \mapsto \dots$.) Discuss solutions to be sure students understand the notation before moving on. For the rest of class or for homework: Problems 6–8.

Day 2: Share answers to Problem 8. You may want to talk about negative scale factors and distortions, or you may not want to focus on these details and instead focus on the ideas of translation and dilation.

The two big ideas in this investigation are translating figures by arithmetic operations on the coordinates of the points and the new notation. The notation is related to the previous investigation on scaling; it will be used frequently throughout the rest of the module, so Problem 8 is particularly important.

ASSESSMENT AND HOMEWORK IDEAS

Any of the problems in this investigation can be used for homework. Assessment problems appear in the next few investigations.

PICTURES FROM RULES,
RULES FROM PICTURES

Materials: graph paper

OVERVIEW

This investigation brings together ideas about translations, dilations, arithmetic operations on the coordinates of figures, and the notation introduced Investigation 5.9. It is the perfect opportunity for assessment. Investigations 5.8 and 5.9 are prerequisite to this one.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Assign Problem 1 for homework the night before beginning the investigation.

Day 1: Students share their solutions to Problem 1. Each student should share at least one solution; solutions for each picture should be presented. (They may need some help with notation and with identifying equivalent rules.) Students then work on Problems 2–6, individually or in small groups, and finish them for homework.

Day 2: Discuss homework, particularly Problems 3–6. In Problems 3 and 4 there are subtle differences between scaling and then translating, or translating and then scaling. How can you tell from the written rule which is done first? In Problem 5, which rules produced squares? Rectangles? Were there any rules that didn't produce a rectangle? As a class or in small groups, read Problem 7 and work on it. Also, have students work on any "Take It Further" or "Write and Reflect" problems that you choose.

Notes: You may want to spend some time talking about "equivalent rules," especially if students in your class generate equivalent but not identical solutions for some of the pictures in Problem 1. For example, show two rules:

- $(x, y) \mapsto (2x + 4, 2y + 6)$
- $(x, y) \mapsto (2(x + 2), 2(y + 3))$.

Ask, "What is the difference in the *process*? What is the difference in the *result*?"

ASSESSMENT AND HOMEWORK IDEAS

Homework is suggested in “Teaching the Investigation” on the previous page.

If students have simply worked straight through this investigation, you may want to give them a short quiz, including one problem like Problem 1 and one like Problem 5. Also add a problem like this:

Without drawing any figures, identify which rules will produce figures congruent to the original, similar to the original, or distortions of the original. How do you know?

- $(x, y) \mapsto (2x + 3, 2y)$
- $(x, y) \mapsto (-x, y - 1)$
- $(x, y) \mapsto (3x + 2, 3y - 2)$
- $(x, y) \mapsto (\frac{x-3}{2}, \frac{y+4}{3})$
- $(x, y) \mapsto (-x, -2y)$
- $(x, y) \mapsto (x, y + 100)$

SCALING POINTS

Materials: graph paper

Technology: Either geometry software or Logo would be helpful for the problems on self-similar figures, but the software is not required.

In this module, we use “scaling a point” as a shorthand for dilating a point with center at the origin.

The day before: Do you need to reserve the computer lab?

OVERVIEW

The main ideas of this investigation are:

- scaling a figure by operating on the coordinates of the points $((x, y) \mapsto \frac{1}{2}(x, y))$;
- scaling a figure with vectors;
- collinearity of all multiples of a point (cP , where c is a number and P is a point);
- an informal introduction to self-similar figures.

Investigations 5.8–5.10 are prerequisites for this investigation. The distance and midpoint formulas (covered in Investigation 5.4) are used in some problems. Similarity is discussed in several problems, but it is relatively informal. If your class hasn’t studied similarity formally, you may want to look over the problems and decide if you need to skip any of them because they would confuse students. An extensive discussion of similarity can be found in the *Connected Geometry* module *A Matter of Scale*.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Day 1: Students read the introduction on their own or aloud in groups and then work on Problems 1 and 2. As a class, discuss the collinearity of all multiples of a point and read about scaling with vectors. Homework: Problems 5–8.

Day 2: Discuss the homework: What happens to the individual points as they are scaled? (They move farther from the origin or closer to it, depending on the scale factor.) What happens to them in relation to the shape they lie on? (They seem to “spread out” or get squeezed together, depending on the scale factor; the shape itself gets either bigger or smaller as we’ve seen before.) Students work on Problems 9–12. End class with a discussion about Problem 12 and the area relationship between similar figures. Homework: Problems 15–17.

Day 3: Discuss the homework and what it takes to “root” a picture that is to be scaled, keeping either a vertex or a whole side in place. The key element, of course, is that the

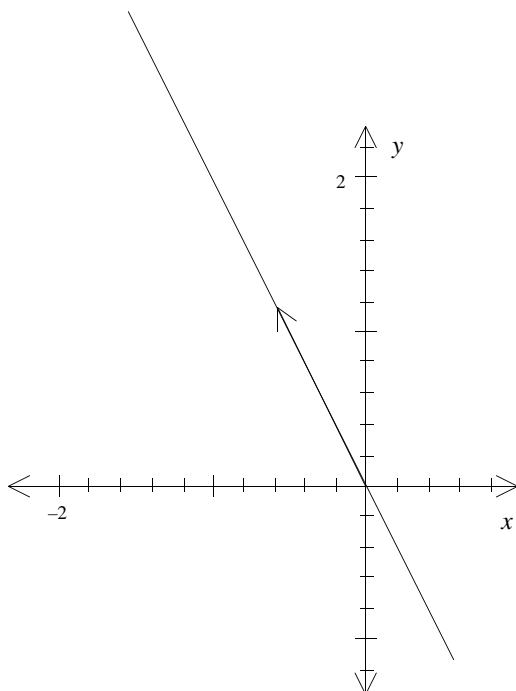
Scaling a point at the origin leaves it in place (the center of dilation doesn't move); scaling points collinear with each other and the origin keeps the new points collinear with each other and the origin.

vertex is at the origin or that the side contains it. Students then work on Problems 13 and 14, in the computer lab if you have access to it. End with a discussion of self-similar figures. The point of the discussion should be to tie ideas of the self-similar figures to the scaling students have been doing. Adding the coordinate axes in Problem 12 lets you notice that scaling picture 1 by $\frac{1}{3}$ gives you exactly the leftmost piece of picture 2, and so on. Without the coordinate axes and the techniques of scaling points in this activity, the relationship might not be as clear.

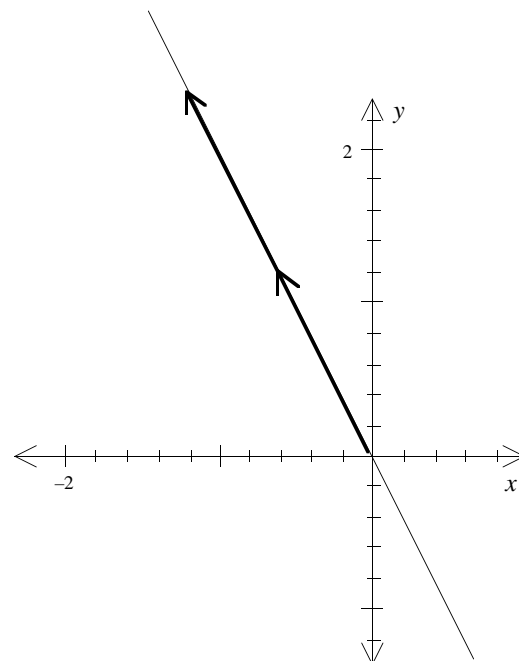
Homework or in-class assessment on Day 4: Problems 18–20, 22, and 23.

Day 4: Discuss the homework. Work on Problem 21 as a whole class. Students work on Problems 24 or 25 in small groups and prepare a solution to present in class at the end of the day or the next day.

Notes: On stretching vectors, it may not be obvious that the following trick doesn't always work. That is, you can't do it for every scalar multiple of a vector. To stretch a vector:



Use a ruler to draw a line through the vector.



Use a compass to mark off a multiple of the length.

But then again, it's not possible to exactly plot irrational multiples of points either.

The *Connected Geometry* module *A Matter of Scale* has a short section on self-similar figures and using Logo to create them.

This will clearly work for all whole number multiples (for negative multiples, you simply mark off the appropriate number of lengths on the opposite side of the origin). For rational multiples of the vector, it is a little trickier. You will have to construct $\frac{1}{2}$ (or $\frac{2}{3}$ or $\frac{5}{18}$) of the length, and then stretch that new vector. For irrational lengths, it may not be possible.

Problems 13 and 14 introduce self-similar pictures using two famous fractals: the Sierpinski triangle (or Sierpinski gasket) and the Koch curve. If your students have never seen fractals before, you may want to take a little time with these problems, talking about scale factors, developing Logo procedures to create them, and talking about what happens at a large number of steps into the sequence. The “Mathematics Connections” section below describes one way to develop the Logo programs as well as some more advanced ideas about self-similar figures.

ASSESSMENT AND HOMEWORK IDEAS

Homework is suggested in “Teaching the Investigation” above. Two assessment opportunities are also suggested.

ADDITIONAL RESOURCES

If you want to pursue fractals further in your classroom, here are some ideas:

1. Abelson and diSessa, *Turtle Geometry*, MIT Press, 1980. This book, among other things, includes methods for developing recursive procedures like the ones suggested in the “Mathematics Connections” section.
2. The *Connected Geometry* module *A Matter of Scale: Pathways to Similarity and Trigonometry* includes a short section on self-similarity.
3. James Gleick’s book *Chaos* (Penguin, USA, 1988) provides a wonderful history of the field and the personalities involved without too much sophisticated mathematics.
4. Mandelbrot’s *The Fractal Geometry of Nature* (W.H. Freeman & Co., 1988) is the classic text, but only parts are accessible to high school students.

5. Ivars Petersen's *The Mathematical Tourist* (W.H. Freeman & Co., 1989) is a great book for the general reader.
6. Clifford Pickover's book *Computers, Pattern, Chaos, and Beauty* (St. Martin's Press, 1991) has some very accessible chapters and some more difficult chapters.
7. National Council of Teachers of Mathematics sells a book called *Fractals for the Classroom* by Peitgen, Jurgens, and Saupe.

MATHEMATICS CONNECTIONS

There are some simple and some difficult ideas behind self-similarity. We suggest that you stick with the simpler ideas for your students. The main idea behind the self-similar figures shown in the Student Module is that, if you look at stage 4 (for example), and cut off one piece of it and scale it up by the right amount, you will get a picture that looks just like stage 3. In general, if you look at stage n , cut off the right piece, and scale by the right factor, you get stage $n - 1$. You can use that idea to build up to recursive procedures for creating the two fractals shown in the Student Module.

For the Sierpinski triangle, start by writing a procedure that will draw the stage 0 triangle:

```
to Tri0 :side
  repeat 3 [ fd :side  rt 120 ]
end
```

For the stage 1, you want to draw one side of the big triangle, then draw one of the smaller triangles inside, and then repeat that two more times. You can use the **Tri0** procedure to draw the smaller triangles:

```
to Tri1 :side
  repeat 3 [ fd :side  rt 120  tri0 :side/2 ]
end
```

The next stage has little stage 1 triangles in the corners, but that's an easy change to make:

```
to Tri2 :side
  repeat 3 [ fd :side  rt 120  tri1 :side/2 ]
end
```

Students should be able to figure out the scale factor of $\frac{1}{2}$, either because they recognize the midlines in triangles in stage 1 or by trial and error.

Notice when tri calls itself, it scales side by $\frac{1}{2}$ and decreases stage by 1.

Students could make as many of these programs as they want, and thereby develop the Sierpinski triangle to any stage they want. However, you can make one procedure that does it all. Notice that each procedure calls the previous one, but they all look exactly the same. The exception is the stage 0 procedure, which stops the process of calling more and more procedures. So the procedure could call itself, with sides scaled by $\frac{1}{2}$, as long as it knows when to stop. You can build in this stop condition by adding a second input, **stage**, that tells you how deep to go in the triangle:

```
to Tri :side :stage
  if :stage=0[stop]
  repeat 3 [fd :side rt 120 tri0 :side/2 :stage-1 ]
end
```

A similar building up of procedures would work for the Koch curve:

```
to Curve0 :length
  fd :length
end

to Curve1 :length
  curve0 :length/3 lt 60 curve0 :length/3 rt 120 curve0 :length/3
  lt 60 curve0 :length/3
end
```

You can find the turning required by realizing that the middle segment is replaced by an equilateral triangle “hat,” so the base angles both need to be 60° .

And so on, with the recursive program looking like this:

```
to Curve :length :stage
  ifelse :stage=0 [fd :length]
  [curve :length/3 :stage-1 lt 60 curve :length/3 :stage-1 rt 120
  curve :length/3
  :stage-1 lt 60 curve :length/3 :stage-1]
end
```

You now have two sequences of shapes. When you pass the sequence to the limit, you get a fractal, a figure that is truly self-similar in that it can be divided into congruent

The “magnification factor” is how much you have to scale those pieces to get back the original shape. It’s the reciprocal of the scale factor we used in the Logo programs.

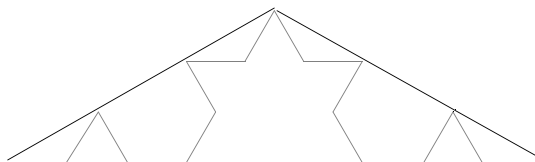
subsets, each of which may be magnified by a constant factor to get the shape itself (not a “previous stage”). There is lots of mathematics here, but it’s subtle. For students, it’s best to think in terms of a sequence of shapes. For you, here are some interesting tidbits:

- **Fractal Dimension:** The most common definition of fractal dimension is

$$D = \frac{\log(\text{number of congruent pieces})}{\log(\text{magnification factor})}.$$

The first of our two examples, the Sierpinski triangle, can be divided into three congruent pieces, each of which must be scaled by 2 to get the original shape, so it has dimension $\frac{\log 3}{\log 2}$, which is less than 2. The Koch curve has four congruent pieces and a scale factor of $\frac{1}{3}$, so its dimension is $\frac{\log 4}{\log 3}$.

- **Perimeter and Area:** The Koch curve has infinite length but contains a finite area (the lower bound is the original segment, the stage 0 curve). One way to see this is to realize that the length at each stage is $\frac{4}{3}$ the length at the previous stage. So the length of the curve (assuming you start with a unit length) is $\lim_{n \rightarrow \infty} (\frac{4}{3})^n$. This does not converge. The area, however, is completely contained by the triangle drawn around the first stage:



In the construction of the Sierpinski triangle, we “remove” the upside-down triangles at each stage, so they are not part of the figure. The limit is a totally disconnected set, so it has no area.

Similarly, the Sierpinski triangle has zero area and infinite perimeter.

For more about these ideas, see any of the books in the “Additional Resources” section of this investigation.

ADDING POINTS

Materials: graph paper

Technology: Geometry software would be useful but is not required.

OVERVIEW

The main ideas in this investigation are:

- translating a figure by adding a point to every point on the figure $((x, y) \mapsto (x, y) + (2, -5))$;
- translating a figure with vectors;
- congruence of translated figures.

This investigation also introduces the “parallelogram rule” for adding vectors, the idea (very informally) of two vectors spanning the plane, and reversing a translation by adding the negative of the point used to translate. Many of the problems here will look familiar; they are nearly the same as the translation problems from Investigation 5.11. Investigations 5.8–5.11 are prerequisites.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Day 1: Students work on Problem 1 individually or in small groups. Have a class discussion on how to find $A + B$ if you don’t know any coordinates. (Methods may include the parallelogram rule if students know it or copying the length and direction of A , but starting at B instead of the origin.) Students then work on Problems 2–8. Conclude class with a discussion about translating shapes by adding a point to each point on the shape. Homework: finish Problems 2–8 and Problems 9–12.

Day 2: Discuss the homework. Talk about reversing a translation (by adding the negative of the point) and the identity transformation (adding the origin to any point). Students work on Problems 16–20 and then discuss the solutions, especially to Problems 19 and 20. Homework: Problems 13–15.

Students may need some help pulling the ideas together. In the discussion, ask questions like:

- Given two vectors A and B , how can you tell if combinations of scalar multiples $aA + bB$ will give you all the points in the plane or just a line?
- Is there any time when these combinations will give you just a point? (This happens only if A and B are both at the origin.)

This is the second example of an identity transformation. In the previous investigation, we saw that scaling by a factor of 1 didn’t change the shape either.

- Would combinations of any two vectors—like $(1, 2, 3)$ and $(3, 7, 0)$ —give you all the possible points in three dimensions (\mathbb{R}^3)?
- If not, how many vectors would you need? What would combinations of the two vectors give you?

Day 3: Read “Ways to Think About It” aloud in class. Ask students to compare how they solved Problems like 1, 5, and 6 to these ideas. What method did they use? Students then work on Problems 21 and 22 and do the geometry software investigation (optional).

You will probably move more quickly through this investigation than through the previous few because students will be more comfortable by now with plotting points, translating, and the $(x, y) \mapsto (\text{blah}, \text{blah})$ notation.

ASSESSMENT AND HOMEWORK IDEAS

Almost all of the problems in this investigation are suitable for homework, allowing for flexibility in how you move through the material.

You may want to finish the investigation with an in-class assessment based on Problems 14, 15, and 21 rather than assigning them as homework or group problems for solving in class.

MAKING THINGS PRECISE

OVERVIEW

Materials: graph paper

This section of the module formalizes notions of vectors and adding and scaling points, digs into the proofs of two theorems about the algebra of points, and ends with students using vectors to prove some standard Euclidean geometric results and to analyze how a pantograph works. It is an ideal exploration for more advanced classes, for classes taking seriously the notion of proof, and for teacher preparation classes.

Students prove theorems about coordinates in this investigation, first by showing that specific points hold true and then by using generic points to prove the theorems. A couple of proofs are given, and students give reasons for each step. Students are also expected to develop their own proofs in some cases.

Familiarity with adding and scaling points is required. The distance formula is also used in some proofs. Congruent triangles are used in Problem 24. You can skip this problem if your class hasn't studied congruence.

What's coming up?
Students will use the theorems developed in this investigation during the next investigation.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Have students read the introduction and complete Problems 1 to 3 the night before you begin the investigation.

Day 1: Discuss the introduction and the answers to Problems 1 to 3. Have individuals complete the exercises in the first “For Discussion” section or complete them orally as a class. Read the “Ways to Think About It” aloud and use the next “For Discussion” section as a class discussion or assign a small group discussion to begin to formulate a theorem. Students should then be able to complete Problems 4–8.

The proof of Theorem 5.3 (“Thing 1” and “Thing 2”) and Problems 9–12 can be assigned as homework.

Day 2: After a discussion of students' work with Theorem 5.3 the night before, hold a class discussion that will lead to a *new and improved* version of Theorem 5.3. This may take some time. If students had difficulty with Problems 9–12, you may want to give them a chance to redo the work after the class discussion. Collect this second version for a grade. Students should then work on Problems 13–15 in preparation for the proof of Theorem 5.4. Assign Problems 16–19 as homework.

Day 3: Discuss homework. Students can work on Problems 20–23 individually or in groups. As a class, discuss the solutions, especially to Problems 20 to 22. Ask if anyone found the point in Problem 23. If not, show them the other point, and assign Problem 24 for homework.

ASSESSMENT AND HOMEWORK IDEAS

Homework is suggested in “Teaching The Investigation” above.

A short quiz would be appropriate at the end of this investigation, and it can vary depending on your goals for the class. Students will be asked to use these particular theorems in the upcoming investigations, so you could preview some of this work:

- Converse of Theorem 5.3:
Suppose A and B are points so that B is collinear with A and the origin. Then there is a number c so that $B = cA$.
 - Check this theorem with several specific pairs of points, but be sure they are collinear with the origin.
 - Prove the theorem with general coordinates.
- Show that the point $P = k(A - B)$ fits the requirement $PB = OA$ and $PA = OB$ when

$$k = \frac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{(a_1^2 - b_1^2) + (a_2^2 - b_2^2)}.$$
 - Check this for several points $A = B$.
 - Prove the theorem with general coordinates.
- If you want to know if a point P is collinear with A and the origin, how would you check? Suppose you want to check if a point P is collinear with two points A and B (rather than A and the origin). Can you devise a test to tell if it is, using the idea from Theorem 5.3 and adding points?
- If A and B are points, then $\overline{O(B - A)} \parallel \overline{AB}$ and $\overline{O(B - A)} \cong \overline{AB}$.
 - Check this for several specific points A and B .
 - Prove the theorem using general coordinates or Theorem 5.4.

USING THE THEOREMS**OVERVIEW**

This investigation relies on Theorems 5.3 and 5.4 presented in the Investigation 5.13. Students use these theorems along with a new theorem to prove several properties of adding and scaling points.

Students should be familiar with Theorems 5.3 and 5.4, the distance formula, and methods for proving that lines are parallel.

TEACHING THE INVESTIGATION

Students should read the introduction and do Problems 1–3 the night before you begin the investigation.

If students have done Problems 1–3 for homework, have them share their answers and methods. Otherwise, have students begin work on Problems 1–4. It may be useful to have pairs work on Problem 4 to brainstorm a method to arrive at the coefficients of A and B . When students are finishing up, move on as a class to Theorem 5.5. Homework can be writing up or finishing Problems 1–4 in addition to other new problems.

The remaining problems in this investigation can be split between homework and classwork. Students are asked to use the distance formula, as well as Theorems 5.3 and 5.4, to prove several things. Problems 8, 9, 12, 13, and 14 may be good class work problems where whole class or small group discussions can take place.

ASSESSMENT AND HOMEWORK IDEAS

Homework suggestions are given in “Teaching the Investigation.” You may want to collect students’ work on some of the very comprehensive problems like Problems 12, 13, or 14 for assessment.

THE ALGEBRA OF POINTS

OVERVIEW

This investigation asks students to prove some properties of points. Familiarity with the Midline Theorem and with calculating midpoints with coordinates is required.

TEACHING THE INVESTIGATION

On the first day, read aloud the introduction to this investigation, as well as Theorem 5.6. Problem 1 asks students to prove the eight parts of Theorem 5.6. You may want to lead students through the first few properties so they know what is expected. They can then work individually or in groups to prove the remaining properties. Ask students to share their proofs of each property. (The proofs should be short, and just one or two proofs for each property should suffice.)

Students can complete Problems 2 and 3 in class or for homework if there is not time. Encourage students to draw pictures to help them generalize the property presented in Problem 3.

ASSESSMENT AND HOMEWORK IDEAS

Problem 4 asks students to use the property that was generalized in Problem 3. As an assessment, ask students to use Problem 3 and the properties in Theorem 5.6 to solve Problem 4 (and Problem 5, especially if your class has a focus on proof). They could also prove it using the distance formula, but they should be encouraged to use the new information.

You may ask students to try the proof with the distance formula as well, to see how much messier it is.

MORE ON SCALING POINTS

OVERVIEW

This investigation introduces a more general form of Theorem 5.3. Students make two generalizations of Theorem 5.3, using their knowledge of square roots and absolute value.

Familiarity with Theorem 5.3 is required. Properties of square roots and the definition of absolute value are reviewed, but students should be comfortable with their use.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Assign the introductory reading and Problem 1 the night before beginning the investigation.

Day 1: Have students explain their proof of Theorem 5.7 (Problem 1) to a classmate. Select one or two students to present the theorem to the whole class, using only their own notes. This activity can be the basis for a class discussion on how to read a proof. Students can then go on to complete Problems 2 and 3 proving other cases of Theorem 5.7. End class with several students presenting their solutions to these problems. Homework: Careful write-up of all three proofs (Problems 1–3) and Problem 4 as well as the reading that follows Theorems 5.8 and 5.9 (ending just before Problem 5) in preparation for the next day’s class discussion.

Day 2: Discuss the reading that follows Theorems 5.8 and 5.9. You may want to spend time making sure students understand the discussion of square roots and absolute values. Have students refer back to Problem 10 and the proof of Theorem 5.3 in Investigation 5.13. Students can complete Problem 5 in class. As a class, read about proving Theorem 5.9.

ASSESSMENT AND HOMEWORK IDEAS

Problems 5 and 6 can be assigned after Day 2. You may want to help students to begin to set up the three conditions.

The “Take It Further” problems may be assigned as homework or completed in class with teacher guidance.

Part of reading a proof is working through the details, being sure you know the reason for moving from one step to the next. You can model this by working through some of the questions about Theorems 5.7 and 5.8.

**MORE ON ADDING
POINTS****OVERVIEW**

Students are asked to fix a glitch in the proof of Theorem 5.4, namely that there are two points for which $PA = OB$ and $PB = OA$.

Investigations 5.13–5.17 are prerequisite. Familiarity with the distance formula and solving systems of equations is required. This investigation requires students to be facile with a lot of algebraic manipulation.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Students should read the introductory material leading up to Problem 1 the night before you begin the investigation.

Day 1: Discuss the problem that students read about the previous night. Students can work on Problems 1, 2, and 3. End class with a discussion about the three problems, the solution presented in the Student Module, and Theorem 5.10. For homework, students do Problems 4–7.

Day 2: Discuss the homework, especially Problem 7, asking students to share their solutions. Discuss “Ways to Think About It” and help students through the proof suggested in Problem 8. Students can examine Theorem 5.11 and work on the problems that follow in class.

ASSESSMENT AND HOMEWORK IDEAS

Homework is suggested in “Teaching the Investigation” above.

The “Write and Reflect” problem at the end of this investigation would make a good written assessment.

You may want to refresh students’ minds regarding solving systems of equations by giving them a couple that can be solved easily by the linear combination “addition” or “elimination” method.

VECTORS AND
GEOMETRY

OVERVIEW

This investigation introduces students to the language and use of vectors. Students are asked to discover and apply several properties of vectors.

Familiarity with coordinates and adding and scaling points is required. Problems 11 and 27 require familiarity with functions and function notation.

TEACHING THE INVESTIGATION

Here is one suggested lesson plan:

Students should read the introductory material and complete Problems 1 and 2 the night before you begin the investigation.

Days 1 and 2: Problems 1–16 deal with properties of vectors and the *HEAD – TAIL* technique of finding the “up-over” for vectors. By the end of this set of problems, students should be familiar with finding equivalent vectors by using the *HEAD – TAIL* rule. You may want to refer back to students’ work on adding and scaling points to show that they were really just using vectors to accomplish these tasks.

Days 3 and 4: In the next group of problems, students move vectors and figures to the origin by performing vector algebra. It is in these problems that the students first see the *HEAD – TAIL* rule written.

In Problem 24 students are also asked to use the *HEAD – TAIL* rule to test relationships between vectors other than equivalence.

The Summary at the end of this investigation provides a good place to stop and review all knowledge of vectors thus far.

ASSESSMENT AND HOMEWORK IDEAS

During days 1 and 2 of this lesson, Problems 7, 8, 11, and 12 are appropriate homework problems.

Each class can begin with a discussion about homework and answering any questions. For most of the class, students will be working on problems. Class can end with a summarizing discussion and information about the homework.

During days 3 and 4 Problems 18–21 and any of the “Checkpoint” problems would be appropriate homework problems. Several problems in the next investigation will be appropriate for final assessment.

**USING VECTORS TO
SOLVE PROBLEMS****OVERVIEW**

This investigation includes a collection of problems that require several skills including adding and scaling points and vector algebra, plus knowledge of several of the theorems pertaining to coordinates and vectors that have been studied in this module.

Familiarity with coordinates, adding and scaling points, vector algebra, and function notation is required.

TEACHING THE INVESTIGATION

This investigation assumes a lot of prior knowledge but introduces no *new* geometry. You can have students work through these problems individually or in pairs and can plan this investigation depending on students' pace through the problems.

Problem 12 asks students to prove the theorem that states that the three medians of a triangle are concurrent at a point that is $\frac{2}{3}$ of the way from any vertex to the midpoint of the opposite side. This result was verified for a specific case in Problem 4 of Investigation 5.15, so you may wish to refer back to that problem. The notes for Problem 12 of the Solution Resource include some interesting properties of the centroid.

ASSESSMENT AND HOMEWORK IDEAS

Appropriate homework problems would include 4, 6, 8, 9, 11, and 12. The Solution Resource provides a nice discussion of Archimedes' discovery of the centroid of a triangle after the suggested solution for Problem 12. If this is assigned as homework, you may want to extend the problem the next day by using the Solution Resource as a point of discussion.

The investigation is designed primarily as a source for final assessment problems. Select problems—perhaps different problems for different students. Students work on the problems in class and for homework and turn in a careful write-up (or do a class presentation). This should take no more than two to three days, depending on how many problems each student is solving.

Suggestions for final assessments:

Exploration of midpoints (Problems 1–7)

Exploration of parallelograms (Problems 1–5, 9 and 10)

Exploration of centers in triangles (Problems 1–4, 11–14)

Pantographs (Problems 1–5 and 15)

A person who answers “zero” to this question might, for example, argue that he or she is correct, since “only 1 line” means “0 different ones.” Casual English often allows such alternative interpretations, and that’s one reason that mathematics, like the legal profession, often uses special vocabulary.

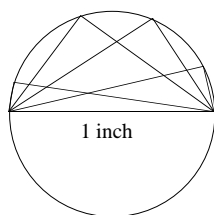
Problem 1 (Student page 1)

- Try “holding” two imaginary points in the air as you picture this situation. By the usual ways of thinking, there can be only one straight line through any two specified points. The wording “how many *different* lines” allows for some argument, but only about the *form* of the answer.
- While people might disagree about the form of the answer to the question in part a, they are less likely to disagree about its substance. That’s not so true for part b. There are at least two interpretations of the word “different” in part b, and they lead to very divergent answers. Both ways of thinking are reasonable, and both are used in geometry.

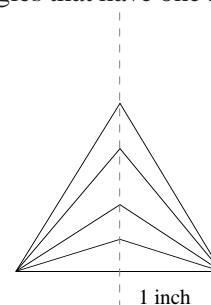
Interpretation 1: In this interpretation, “different” refers only to the size of the square and to the size and shape of the triangle, not to their placement or orientation. If you specify the length of a square’s side (or its diagonal), you know “everything there is to know” about the square. From this point of view, only two squares can use the two points given in Problem 1 as vertices: one with the two points defining a side and one with the two points defining a diagonal. Only one square can have a sidelength of 1”.

Interpretation 2: On the other hand, it may be important to realize that lots of “different” squares can be made, all with the same sidelength: some might be drawn on the wall, others on the table, some tilted, others level, some nearby, others far away. In this interpretation, the two points in space limit the square to two *sizes* (sidelength or diagonal length) but only partially limit the placement. Squares through those points can lie in different planes, so there is an infinite number of them.

Problem 2 (Student page 1) Using Interpretation 1 above, there is only one square with sidelength 1”. That is not true of a triangle. If you specify the length of one side of a triangle, there are still infinitely many *different* triangles that have one side of that specified length.



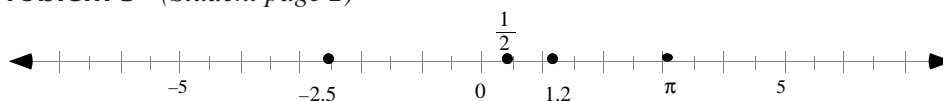
right triangles



isosceles triangles

Using Interpretation 2, there are infinitely many of both squares and triangles.

Problem 3 (Student page 2)



Problem 4 (Student page 2) One way is to specify the distance along the path from the origin. Any units could be used, but a unit must be specified.

Problem 5 (Student page 3) One way to assign coordinates to points in this space is to modify whatever idea you used successfully for Space 2. This space seems to require two pieces of information. Here, in addition to saying “where along the path,” you must also specify the path.

The paths have not been given numbers, but you might decide that the path containing *A* is path 1, *B* path 2, and so on.

How you choose to assign coordinates depends on your goals. If you had a reason to do so, you could “pack” those two pieces of information in some clever way that makes them *look* like just one piece of information. For example, the number 1.62 might be used to mean “on path 1 and 62% of the way from the origin to the end.”

Problem 6 (Student page 3) Here’s one of many possible solutions:

Starting with the origin and measuring clockwise, assign numbers indicating how far along the circle one must go in order to reach the point. That distance could be expressed in absolute distance (with some units), as a fraction of the way around the circle (numbers 0 to 1), or even as degrees of arc covered (0 to 360). Interestingly, this seems to assign each point many coordinates! The coordinates 0 and 1, for example, both point to the origin (as do 2, 3, 4, and all other integers). If *A* is at 120° , then it is also at 480° (that is, $360 + 120$), 840° (that is, $360 + 360 + 120$), and so on. If *B* is at $\frac{2}{3}$, it is also at $\frac{5}{3}$, $\frac{8}{3}$, $\frac{11}{3}$, and so on.

Problem 7 (Student page 4) The number of dimensions of a space is an important factor in how we choose to assign coordinates (addresses) to the points in the space. Perhaps most often we assign to a coordinate system the same number of dimensions as the space it’s meant to represent. But how we choose to assign coordinates to points in a space also depends on our purposes. So, the dimensions of a coordinate system can be different from the dimensions of the space.

But people (including mathematicians) tend to be casual about specifying the dimension of a figure in space. For example, most of the time, context is enough to tell us

whether we are discussing a planar region (two-dimensional) or its boundary (a bent line), so we don't bother adding information about the dimensions. It is equally acceptable to talk about the "area of a circle" and the "circumference of a circle," even though what is meant by "circle" seems to change dimensions: the words *area* and *circumference* (or *perimeter*) tell us whether it's the *region inside* or the *boundary* that is intended.

In a coordinate system, the "dimension" is the number of independent pieces of information that is being supplied.

Problem 8 (*Student page 6*) A one-dimensional system will still work. You can assign each point on the circle the "compass heading" from C (the degrees of rotation clockwise from the north, for example). Is C then the "origin"? That might make sense from some perspectives, but it might feel more consistent with this scheme to think of the "north point" as the origin (the distinguished point) and "clockwise" as the distinguished direction. Then the origin is part of the space (which seems nice), and C , along with the notion of using degrees, is more like a part of the "scale"—the way one does the measuring—than like a true starting point, or zero or origin.

Two-dimensional Cartesian coordinates—an ordered pair of numbers indicating horizontal location and vertical location with respect to C —would also uniquely identify all the points on the circle, but they address too much: all the points in the plane, inside and outside the circle.

LOCATING POINTS IN TWO DIMENSIONS

Problem 1 (Student page 8) On paper, it would seem most natural to specify the row, starting at the top, and then to specify the position within the row. That is also a common scheme in locating characters in a computer window.

9th row, 11th character
which might be written as
[9 11]

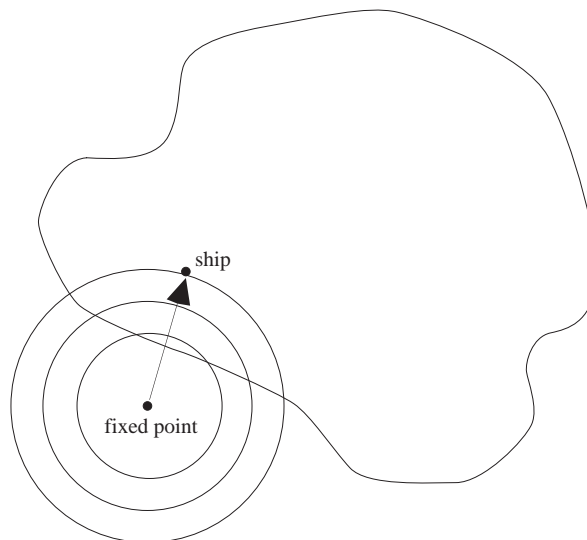
This is a line of text, but it seems far too small to be of any real use, so I might have to do something else. On paper, it would seem most natural to specify the row, starting at the top, and then to specify the position within the row. That is also commonly used on the computer.

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Problem 2 (Student page 8) It seems easiest to specify compass direction and distance. That is, in fact, the information that a radar screen reports. The radar antenna rotates around in a circle, and reports back the distance of the things it “sees.”

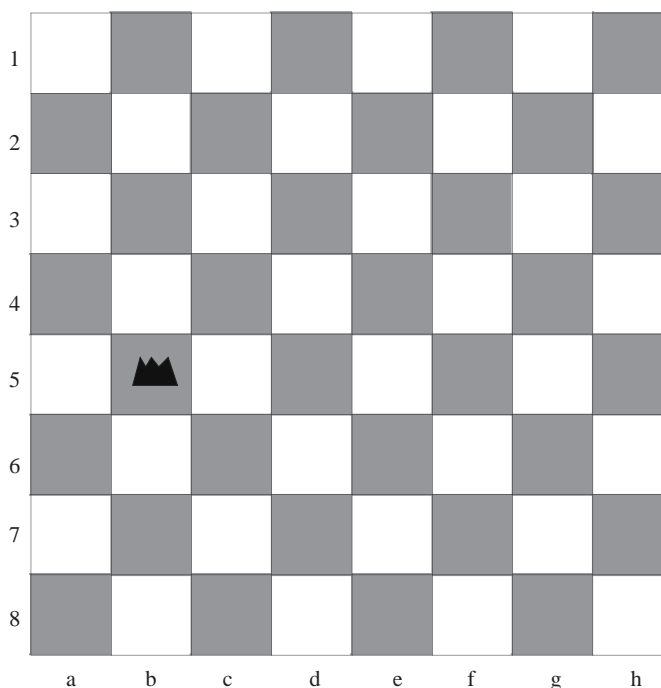


Problem 3 (Student page 8) The chessboard is finite, and so a one-dimensional coordinate system—for example, numbering the squares from 1 to 64 in some orderly way—is convenient enough. There are many reasonable two-dimensional schemes: for example, letters and numbers could be used for the columns and rows, or something

The bottom of the board is the side where the white pieces start. The king shown is on space b5 (the letter always comes first).

more like the Cartesian system could be used. Because players sit on opposite sides of the board, it might be best to design a system that is convenient for both players.

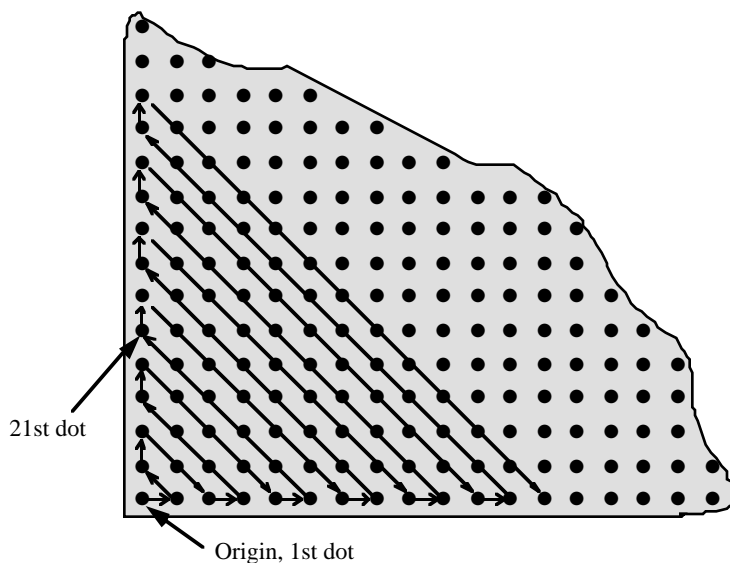
The common system for recording chess games uses letters along the bottom of the board and numbers along the left side, as in this picture:



Problem 4 (Student page 8)

- a. Starting with the left-hand column, number the columns 1, 2, 3, ...; then starting with the bottom row, number the rows 1, 2, 3, The origin would then be assigned the coordinates (1, 1).
- b. The mathematician Georg Cantor invented a one-dimensional coordinate scheme to show that the rational numbers could be “counted”—that is, that every single rational number could be matched with a counting number.

The following illustration shows how Cantor’s scheme works. Starting at the origin, and following the arrows, each dot is counted: 1, 2, 3, 4,



Cantor’s “dots” were the rational numbers, arranged essentially like this:

$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$	
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	
$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$	
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	
$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$	
$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$	$\frac{8}{1}$

Every possible rational number is included in such an arrangement (extended infinitely, as the dots were). Many, of course, are listed more than once, but even with these

This means that there is “the same number” of counting numbers and rationals. That is, the sets \mathbb{N} (the counting numbers) and \mathbb{Q} (the rational numbers) have the same cardinality, which we call \aleph_0 (“aleph-null”). This is a somewhat counterintuitive result!

duplications, Cantor was able to show that each element in this array could be assigned its own counting number without ever running out of counting numbers.

Problem 5 (Student page 9) Many schemes will work. The Cartesian scheme lists, in order, two distances from the origin: the distance to the right of the origin and distance above it.

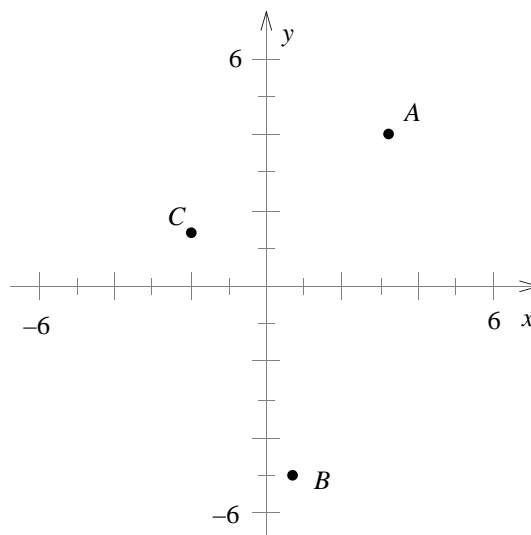
Problem 6 (Student page 11)

- a. Coordinates that are not integers must be estimated. Your answers may vary slightly from those given in the following table.

Point	Coordinates
<i>E</i>	$(-\frac{1}{2}, 4)$
<i>O</i>	$(0, 0)$
<i>I</i>	$(0, 1)$
<i>C</i>	$(3, 2)$
<i>J</i>	$(0, -1)$
<i>H</i>	$(2.5, -2.5)$
<i>M</i>	$(-1, 1\frac{3}{4})$
<i>L</i>	$(-2.5, -2.5)$
<i>D</i>	$(-1\frac{3}{4}, 1)$
<i>B</i>	$(2, 3)$
<i>G</i>	$(-\frac{1}{2}, -3)$
<i>K</i>	$(1, 0)$
<i>A</i>	$(-1, 0)$
<i>F</i>	$(-1, -2)$

- b. Point *N* would be “southeast” of point *H*, both to the right of it and below it.
- c. If point *P* is directly above point *F*, then the *y*-coordinate of *P* would be greater than the *y*-coordinate of *F*, but the *x*-coordinates would be the same.

Problem 7 (Student page 11) The line through both points in the first pair is parallel to the *x*-axis because the two points have the same *y*-coordinates. For the same reason, the line through the points in the second pair is parallel to the line through the other pair of points. Within each pair, the two points are equidistant from the origin, and you get either point by reflecting the other over the *y*-axis.

Problem 8 (Student page 11)

Problem 9 (Student page 12) The next two coordinate pairs in the sequence are (7, 1) and (9, 1); the preceding pairs are (0, 0) and (0, 1).

Problem 10 (Student page 12) If the values of x and y of point $A = (x, y)$ are both positive, then:

- Point $B = (-x, y)$ is in quadrant II;
- Point $C = (-x, -y)$ is in quadrant III;
- Point $D = (x, -y)$ is in quadrant IV;
- Figure $ABCD$ is a rectangle.
- Except for their rectangularity, the figures produced by different choices of A may all be different. However, they will all be oriented with sides parallel to the x - and y -axes.

Problem 11 (Student page 12)

Why did the authors bother to write “appear to have integer coordinates” and “If they do...?”

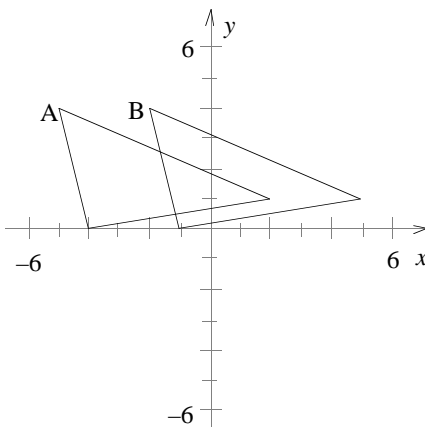
- The vertices of the square appear to have integer coordinates. If they do, then $A = (4, 2)$, $B = (-2, 4)$, $C = (-4, -2)$, and $D = (2, -4)$.

- b.** The other two coordinates, in order, are $(3, -1)$ and $(-1, -3)$.
- c.** Answers will vary, but they should follow the pattern described in the answer for part d.
- d.** In general, if a square (level or nonlevel) is centered on the origin and the coordinates of one vertex are (x, y) , then the other three vertices are at $(-y, x)$, $(-x, -y)$, and $(y, -x)$.

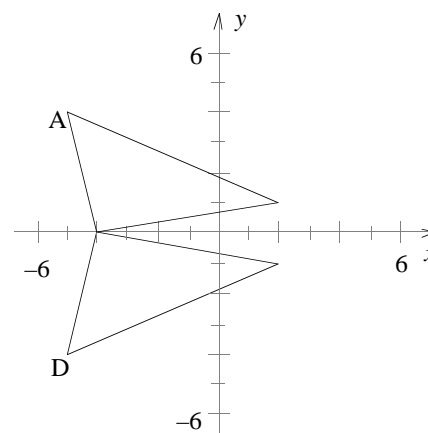
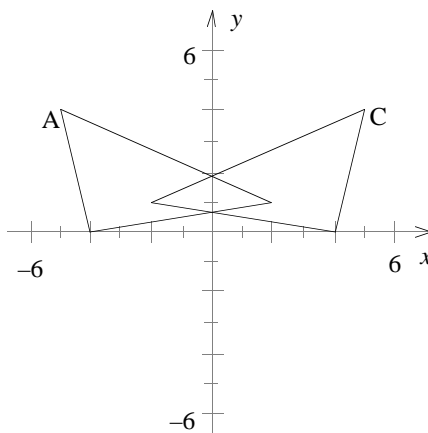
Problem 12 (Student page 12)**a.**

A	B	C	D	E	F	G
(x, y)	$(x + 3, y)$	$(-x, y)$	$(x, -y)$	$(2x, 2y)$	$(\frac{x}{2}, \frac{y}{2})$	$(-y, x)$
$(2, 1)$	$(5, 1)$	$(-2, 1)$	$(2, -1)$	$(4, 2)$	$(1, \frac{1}{2})$	$(-1, 2)$
$(-4, 0)$	$(-1, 0)$	$(4, 0)$	$(-4, 0)$	$(-8, 0)$	$(-2, 0)$	$(0, -4)$
$(-5, 4)$	$(-2, 4)$	$(5, 4)$	$(-5, -4)$	$(-10, 8)$	$(-2.5, 2)$	$(-4, -5)$

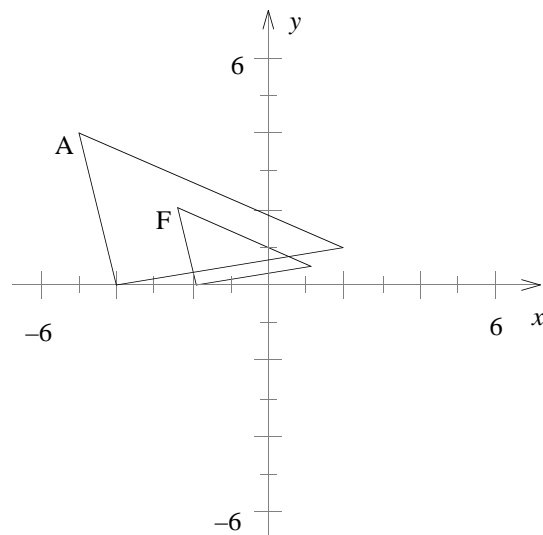
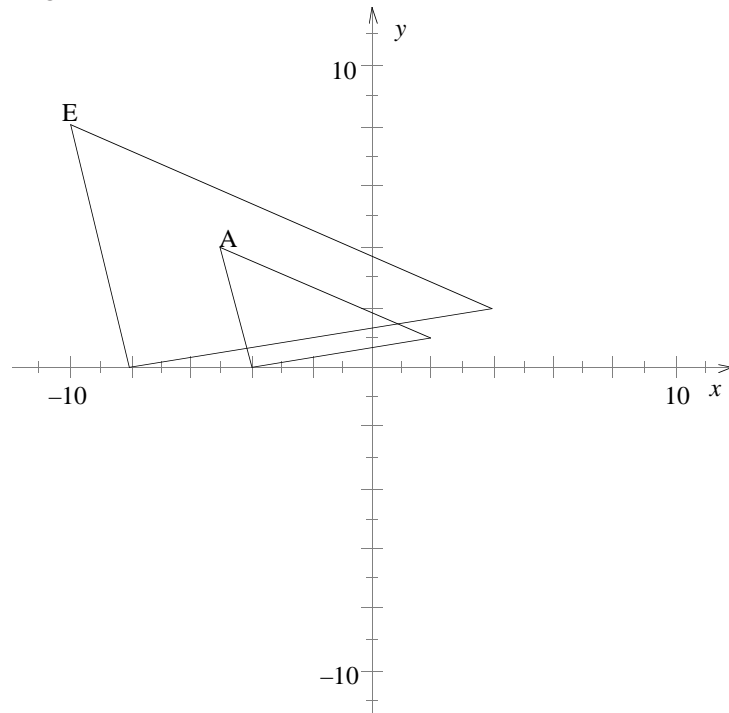
- b.** The triangle described by the three points in column B is identical (congruent) to the triangle described by the three points in A, but translated three units to the right.



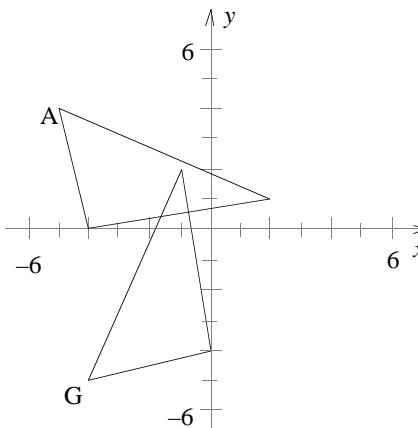
- c.** Triangle C is a reflection of triangle A through the y -axis. Triangle D is a reflection of triangle A through the x -axis.



- d.** Triangles E and F are scaled copies of triangle A. They are similar to triangle A (the same shape as A), but twice and half its size, respectively. The vertices of triangle E are twice as far from the origin as the vertices of triangle A, and the vertices of triangle F are half as far from the origin as the vertices of triangle A.



Triangle G is congruent to triangle A, but rotated 90° counterclockwise about the origin.



Problem 13 (Student page 13)

- a. If both coordinates are positive, the point is in quadrant I.
- b. If the x -coordinate is negative and the y -coordinate is positive, the point is in quadrant II.
- c. If both coordinates are negative, the point is in quadrant III.
- d. If the x -coordinate is positive and the y -coordinate is negative, the point is in quadrant IV.

Problem 14 (Student page 13)

- a. The statement is true.
- b. When E is directly to the right of F , the y -coordinates will be equal.
- c. When E is directly below F , the x -coordinates will be equal and the y -coordinate of E will be less than that of F .

Problem 15 (Student page 14) Because tick marks are not shown on the graphs, some estimation is involved in finding the coordinates in all parts of this problem. Therefore, your answers may vary slightly from those given. This applies to the x -coordinates in part a and both coordinates in part b.

- a.** For this figure the coordinates appear to be $(-7, -2)$, $(-4, 1)$, $(-1, -2)$, $(2, 1)$, $(3, 0)$, $(4, 1)$, $(7, -2)$, $(8, -1)$, and $(9, -2)$. Note that the first, third, seventh and ninth points (starting from the left) share the same y -coordinate, as do the second, fourth, and sixth points. Also, the second point has an x -coordinate that is halfway between those of the first and third points.
- b.** We have the same figure as in part a, but the coordinates shown are twice the coordinates for the first figure. So the coordinates for the figure here should be twice those found in part a.
- c.** For this figure, the same kinds of things are important: there are several pairs of y -coordinates that should be the same, and there are four points that share the same x -coordinate. The points are $(4, -8)$, $(-3, -8)$, $(-3, 1)$, $(-2, 2)$, $(-3, 3)$, $(-3, 5)$, $(7, 5)$, $(5, 3)$, and $(10, 1)$. (There's certainly room for interpretation of some of these. For example the y -coordinate of the first two points listed might be -7 or -9 instead.)

Problem 16 (Student page 15)

- a.** D 's x -coordinate will be less than C 's, and the y -coordinates will be equal.
- b.** The x -coordinates will be equal, and D 's y -coordinate will be greater than C 's.
- c.** D 's x -coordinate will be greater than C 's, and C 's y -coordinate will be greater than D 's.

Problem 17 (Student page 15)

- a.** Quadrants II and III are shaded.
- b.** Quadrants I and II are shaded.
- c.** Quadrant II is shaded.

Problem 18 (Student page 16)

- a.** Quadrant II
- b.** Quadrant I
- c.** Quadrant IV

Problem 19 (Student page 16)

- a.** The only vertical line through the origin is the y -axis. It is a boundary of all four quadrants, but doesn't really pass *through* any.
- b.** Any other vertical line must pass through exactly two quadrants.
- c.** A slant line through the origin must pass through exactly two quadrants.
- d.** Any other slant line must pass through exactly three quadrants.

Problem 20 (Student page 16) Any line other than the axes must pass through at least two quadrants because at least one of the two coordinates of the points on the line is changing, and must eventually assume both positive and negative values. No line can pass through all four quadrants. A slanting line that passes through only two quadrants *must* pass through the origin.

Problem 21 (Student page 17)**a.**

Point	Coordinates
E	$(6, 270^\circ)$
H	$(3, 240^\circ)$
C	$(4, 180^\circ)$
J	$(3\frac{1}{2}, 320^\circ)$
D	$(8, 210^\circ)$
B	$(3\frac{1}{2}, 90^\circ)$
G	$(5, 135^\circ)$
I	$(1, 90^\circ)$
A	$(4, 0^\circ)$
F	$(7, 45^\circ)$

- b.** Infinitely many points can have coordinates $(4, a^\circ)$. These lie along the circle that is four units from the “pole” (the circle with its center at the pole and passing through points A and C).
- c.** Infinitely many points can have coordinates $(r, 60^\circ)$. These lie along the ray marked 60° .
- d.** Even though a can vary, only one point (the pole), has coordinates $(0, a^\circ)$.

LINES, MIDPOINTS, AND DISTANCE

Problem 1 (*Student page 20*) On the y -axis, the value of the x -coordinate is always zero. It is also true that if the x -coordinate of a point is 0, the point is on the y -axis.

Problem 2 (*Student page 20*) Two points on the line are $(213, 1)$ and $(-17, 1)$. Any points with a y -coordinate of 1 will be on this line.

Problem 3 (*Student page 20*) For parts a and b, answers will vary. The answers given here are just examples.

- a. $(3, 700)$, $(3, 456)$, $(3, 23)$, $(3, 0)$, $(3, -13)$, and $(3, -59)$ are on ℓ .
- b. $(4, 700)$, $(4, 7)$, $(5, 23)$, $(5, 0)$, $(-3, 3)$, and $(2, 3)$ are not on ℓ .
- c. A point will lie on ℓ if its x -coordinate (or first coordinate) is 3.
- d. Take any vertical line; all its points will share the same x -coordinate.

Problem 4 (*Student page 20*) For parts a and b, answers will vary. The answers given here are just examples.

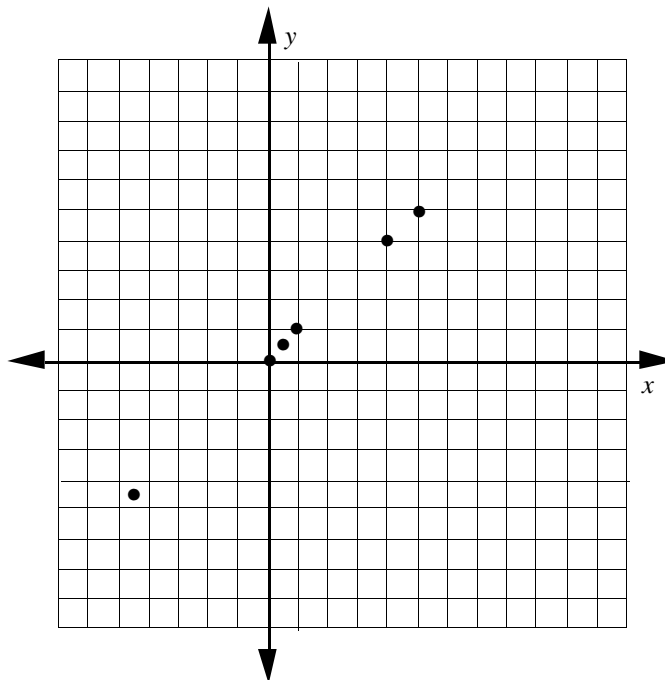
- a. $(13, 7)$, $(4, 7)$, $(5, 7)$, $(0, 7)$, $(-45, 7)$, and $(-1, 7)$ are on m .
- b. $(4, 700)$, $(-5, 23)$, $(5, 0)$, $(-3, 3)$, $(13, -6)$, and $(4, -8)$ are not on m .
- c. A point will lie on horizontal line m if its y -coordinate is 7.

Problem 5 (*Student page 20*) If B is on the same horizontal line as A , where $A = (s, t)$, B will have the same y -coordinate as A , so we know its y -coordinate is t .

Problem 6 (*Student page 20*) The y -coordinate must be 2 and the x -coordinate -4 , so the point of intersection is $(-4, 2)$.

Problem 7 (Student page 21)

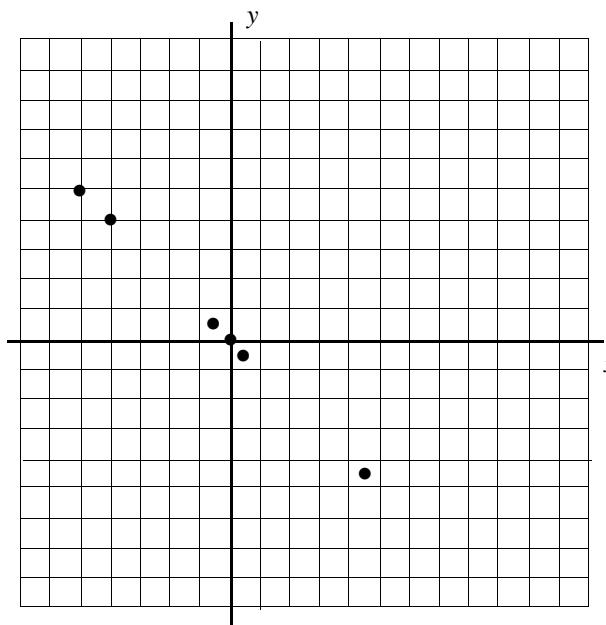
- a. Answers will vary. Six such points are $(1, 1)$, $(4, 4)$, $(5, 5)$, $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, and $(-4.5, -4.5)$.



- b. The graph will be a line through the origin and the first and third quadrants.

Problem 8 (Student page 21)

- a. Answers will vary. Six such points are $(-1, 1)$, $(-4, 4)$, $(-5, 5)$, $(0, 0)$, $(\frac{1}{2}, -\frac{1}{2})$, and $(4.5, -4.5)$



- b. The graph will be a line through the origin and the second and fourth quadrants.

Problem 9 (Student page 21) The lines intersect at the point $(12, 12)$.

Problem 10 (Student page 21) For parts a and b, answers will vary. The answers given here are just examples.

- a. $(-100, -1.95)$, $(-4, -1)$, $(-1, 3.99)$, $(0, 0)$, $(\sqrt{3}, 3)$, $(10, -1.8)$
 b. $(-100, -2)$, $(-4, -14)$, $(-1, 4.5)$, $(0, 4)$, $(4, 100)$, $(10, -2.1)$
 c. A point will be between ℓ and m if $-2 < y < 4$.

Problem 11 (Student page 22) For parts a and b, answers will vary. The answers given here are just examples.

- a. $(2.01, 4.9), (6, 1), (3, -5.9), (7.999, -5.999)$
- b. $(0, 0), (15, 1), (1.999, 4), (\sqrt{65}, 0)$
- c. $(\sqrt{5}, -6), (2, 0), (2, 1.867), (\pi, 5)$
- d. A point will lie inside the rectangle if $2 < x < 8$ and $-6 < y < 5$.

Problem 12 (Student page 22)

- a. This will be the region between and including the vertical lines $x = 5$ and $x = 8$.
- b. This will be the region between and including the horizontal lines $y = -3$ and $y = 6$.
- c. There is an overlap, in the shape of a rectangular region, with vertices $(5, -3), (5, 6), (8, 6), (8, -3)$. Its area is 27 square units.
- d. There are many ways to change the instructions to get a square—but they all involve making the distance between the vertical lines the same as the distance between the horizontal lines. Change part a to $5 \leq x \leq 14$ or $-1 \leq x \leq 8$, or change part b to $-3 \leq y \leq 0$ or $3 \leq y \leq 6$. Or come up with ways to change both parts a and b so that the distance between the vertical lines equals the distance between the horizontal lines.

Problem 13 (Student page 22) This problem is tricky. Careful drawings on graph paper will be helpful. The quadrilateral with vertices $(4, 3), (4, 6), (-2, 5)$, and $(-1, 2)$ has no parallel sides, so it is neither a trapezoid nor a parallelogram. Therefore, statements c and f are false.

However, the problem does not require that the four given points be the vertices, but only that one of its sides passes through $(4, 3)$ and $(4, 6)$, while another side passes through $(-2, 5)$ and $(-1, 2)$. Also, the problem does not specify whether the two sides containing the given points are opposite or adjacent sides of the quadrilateral. Because the segments connecting the four given points can be extended to any length, there are infinitely many quadrilaterals that satisfy the conditions given in the problem. We need to determine whether any of these figures are parallelograms or trapezoids.

If the side containing $(4, 3)$ and $(4, 6)$ is opposite the side containing $(-2, 5)$ and $(-1, 2)$, we can never have either pair of opposite sides parallel, so the quadrilateral can never be a trapezoid or a parallelogram. However, we can form quadrilaterals in which the two specified sides are adjacent sides in the following way: Draw the vertical line through $(4, 3)$ and $(4, 6)$ and the line through $(-2, 5)$ and $(-1, 2)$. These two lines intersect at the point $(4, -13)$. Now draw a line through $(4, 6)$ parallel to the line

through $(-2, 5)$ and $(-1, 2)$, and also draw the vertical line through $(-2, 5)$. These two lines intersect at the point $(-2, 24)$. Label the following points $A = (-2, 24)$, $B = (4, 6)$, $C = (4, -13)$, and $D = (-2, 5)$. Look at quadrilateral $ABCD$. Two of its sides are parallel because they are segments of vertical lines, and all vertical lines are parallel. The other two sides were constructed to be parallel. Thus, $ABCD$ is a parallelogram. Although the given points $(4, 3)$ and $(-1, 2)$ are not vertices of this parallelogram, they lie on the appropriate sides of the parallelogram. Since we have found one parallelogram that meets the requirements of the problem, we know that statement d is false and statement e is true.

$ABCD$ is also a trapezoid, since every parallelogram is a trapezoid. There are also infinitely many trapezoids with exactly one pair of parallel sides that meet the requirements of the problem. To find them, extend the side \overline{CD} further upward into quadrant II. Choose a point on this line such as $(-4, 11)$. Draw any segment connecting this point to side \overline{AB} . We have now formed a trapezoid that goes through the four required points and has exactly one pair of opposite sides parallel. We have seen that some, but not all, quadrilaterals that meet the requirements of the problem are trapezoids. Therefore, statement a is false, and statement b is true.

In summary, statements b and e are true; all the others are false.

Problem 14 (Student page 23)

- a. $AB = 2$
- b. $M_{\overline{AB}} = (8, 5)$

Problem 15 (Student page 23)

- a. $AB = 391$
- b. $M_{\overline{AB}} = (2, 200.5)$

Problem 16 (Student page 23)

- a. $CD = 17$
- b. $M_{\overline{CD}} = (3.5, -7)$
- c. One conjecture might be something like this: “If two points lie on the same horizontal or the same vertical line, they will share one coordinate, as will their midpoint. To find the other coordinate of the midpoint of two such points, find the average of the two nonequal coordinates (or as some students say ‘split the difference’ between them).”

$M_{\overline{AB}}$ represents the midpoint of \overline{AB} . We will use this notation for midpoints throughout the solutions in this module.

Problem 17 (Student page 23)

- a. $IJ = 30$
- b. $KL = 20$
- c. $MN = 183$

Problem 18 (Student page 23) Plot the four given points on graph paper. Form a right triangle with A and B as endpoints of the hypotenuse. The third vertex can be either $(8, 2)$ or $(4, 5)$. The lengths of the legs of such a right triangle are 3 and 4, so $AB = 5$ by the Pythagorean Theorem. Now draw a right triangle with C and D as endpoints of the hypotenuse. The third vertex can be either $(-7, 3)$ or $(-4, 7)$. Again, we have a 3–4–5 triangle. Since $AB = CD = 5$, we have shown that \overline{AB} and \overline{CD} are definitely congruent.

A slightly different way to reach the same conclusion uses the same right triangles, but does not require calculating the lengths of the hypotenuses: Let $E = (8, 2)$ and $F = (-7, 3)$. Compare $\triangle ABE$ and $\triangle DCF$. Since $AE = 4$ and $DF = 4$, we have $\overline{AE} \cong \overline{DF}$. Since $BE = 3$ and $CF = 3$, we have $\overline{BE} \cong \overline{CF}$. We also have $\angle E \cong \angle F$ because all right angles are congruent. Therefore, $\triangle ABE \cong \triangle DCF$ by the SAS congruence postulate, and, consequently, $\overline{AB} \cong \overline{DC}$ by CPCTC (corresponding parts of congruent triangles are congruent). Since \overline{DC} and \overline{CD} are the same segment, we have again shown that \overline{AB} and \overline{CD} are definitely congruent.

Problem 19 (Student page 24)

- a. $PQ = |c - b|$
- b. $PQ = |b - a|$

Problem 20 (Student page 24) \overline{OB} is the diagonal of a square of sidelength 1. By the Pythagorean Theorem, we have $1^2 + 1^2 = \overline{OB}^2 = (\sqrt{2})^2$. The length of the diagonal is $\sqrt{2}$.

Problem 21 (Student page 24)

- a. $EF = 3$; $FG = 4$
- b. $EG = 5$

Problem 22 (Student page 25)

- a. $A = (5, 5)$ and $B = (5, 0)$
- b. $AB = 5$
- c. $\text{Area} = \frac{1}{2}(5)(5) = 12.5$
- d. $AO = 5\sqrt{2}$; this is the same as the triangle and hypotenuse \overline{OB} in Problem 20, only scaled up by 5.

Problem 23 (Student page 25)

- a. Eight such points are $(5, 0)$, $(0, -5)$, $(3, 4)$, $(-3, 4)$, $(4, 3)$, $(-4, 3)$, $(-4, -3)$, and $(0, 5)$. Other answers are possible.
- b. The picture is a circle with radius 5. All points on the circle are 5 units from the origin.
- c. Any right triangle in which the sum of the squares of the lengths of the legs is 5^2 will help us find more coordinates. Try $1^2 + y^2 = 5^2$ to find the points $(1, 2\sqrt{6})$, $(-1, 2\sqrt{6})$, $(-1, -2\sqrt{6})$, $(1, -2\sqrt{6})$, $(2\sqrt{6}, 1)$, and so on for eight points, or $(\sqrt{2})^2 + y^2 = 5^2$ for $(2, \sqrt{23})$, $(-2, \sqrt{23})$, and so on.

Problem 24 (Student page 25) Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be any two points which do not lie on the same horizontal or vertical line. (This will work for any choice of coordinates in which $x_1 \neq x_2$ and $y_1 \neq y_2$.) Form a right triangle in which P and Q are endpoints of the hypotenuse and the right angle vertex is at $R = (x_2, y_1)$. Apply the Pythagorean Theorem to $\triangle PQR$ to find the length of the hypotenuse: Subtract the x -coordinates and square the difference; subtract the y -coordinates and square the difference; add these two results and take the square root:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Recall that the problem asked you how to find the distance between *any* two points if you know their coordinates. If the two points lie on either the same horizontal or vertical line, you will not be able to form a right triangle as described above. If there is no right triangle, the Pythagorean Theorem does not apply. However, it turns out to be easier to find the distance between two points when they lie on the same horizontal or vertical line than when they don't: Simply take the absolute value of the difference between the nonequal coordinates: The distance between (x_1, y_1) and (x_1, y_2) is $|y_2 - y_1|$, and the distance between (x_1, y_1) and (x_2, y_1) is $|x_2 - x_1|$. Why do we need the absolute value bars? They make sure that the distance between any two points is never negative. (In the distance formula given above, we will never get

a negative distance because the differences between coordinates are squared.)

In fact, the distance formula also applies in the special cases we have just discussed:

If $x_1 = x_2$, we get

$$d = \sqrt{0^2 + (y_2 - y_1)^2} = |y_2 - y_1|,$$

and if $y_1 = y_2$, we get

$$d = \sqrt{(x_2 - x_1)^2 + 0^2} = |x_2 - x_1|.$$

What if $x_1 = x_2$ and $y_1 = y_2$? Then the “two points” are same point. Here, the formula still works:

$$d = \sqrt{0^2 + 0^2} = 0.$$

So our distance formula works in every possible case.

Problem 25 (Student page 25) Using the formula above,

$$\begin{aligned} d &= \sqrt{(6 - 1)^2 + (15 - 3)^2} = \sqrt{5^2 + 12^2} \\ &= \sqrt{25 + 144} = \sqrt{169} = 13. \end{aligned}$$

Problem 26 (Student page 26) It may be easy to use the grid to see that $M = (1, 1)$, but the next problem doesn't provide the same kind of situation. Some other method is needed. Unless you know another way, you might find the midpoint in the horizontal direction separately from the midpoint in the vertical direction to find first the x -coordinate and then the y -coordinate.

Problem 27 (Student page 26)

$$M_{\overline{AB}} = (-3, -0.5)$$

$$M_{\overline{CD}} = (2.5, 0)$$

$$M_{\overline{AD}} = (-0.5, -3)$$

$$M_{\overline{BC}} = (0, 2.5)$$

$$M_{\overline{AC}} = (0, 0)$$

$$M_{\overline{BD}} = (-0.5, -0.5)$$

Problem 29 (Student page 27) Some students have written formulas like this: if x is less than w , then the x -coordinate of $M_{\overline{GH}} = x + \frac{|x-w|}{2}$; otherwise the x -coordinate of $M_{\overline{GH}} = w + \frac{|x-w|}{2}$. And if y is less than z , then the y -coordinate of $M_{\overline{GH}} = y + \frac{|y-z|}{2}$; otherwise it's $z + \frac{|y-z|}{2}$.

Another formula is $M_{\overline{GH}} = (\frac{x+w}{2}, \frac{y+z}{2})$, which is much easier to use, but it's not always clear to students why this works. This will be explored in the next investigation.

Problem 30 (Student page 27) The coordinates of the midpoint may be written in decimal or fraction form:

$$M_{\overline{JK}} = (-0.5, 3.5)$$

or

$$M_{\overline{JK}} = \left(-\frac{1}{2}, \frac{7}{2}\right).$$

Problem 31 (Student page 27)

The equation $x = -2$ describes this line.

- There is only one vertical line through $(-2, 3)$. (All points on the line have the x -coordinate -2 , while the y -coordinates of these points may vary.)
- The y -coordinate of every point on the horizontal line through $(3, -5)$ is -5 . The x -coordinate of every point on the vertical line through $(-1, 9)$ is -1 . The only point on both of these lines, that is, their point of intersection, is $(-1, -5)$.

Problem 32 (Student page 27)

- Infinitely many lines are parallel to any given line.
- Because $(7, 4)$ is on m one unit to the right of $(6, 4)$ on ℓ and because the two lines are parallel, we can look one unit to the right of the other known point on ℓ , $(-3, 1)$, and be sure that $(-2, 1)$ is on m .
- We can use the same reasoning again but with two translations. Point $(8, 3)$ is two units to the right of and one unit below $(6, 4)$. The same two translations of $(-3, 1)$ yield $(-1, 0)$.

Problem 33 (Student page 27) Answers will vary; these are examples:

- $(2, 4)$, $(5, 4)$, $(100, 4)$, $(-4, 4)$;
- $(4, 4)$, $(4, -4)$, $(4, 100)$.

Problem 34 (Student page 27) $E = (113, 19)$; $D = (116, 15)$.

Problem 35 (Student page 28) There are many possibilities; these are just examples:

- a. $(21, 7)$ and $(46, 7)$;
- b. $(-16, -13)$ and $(-16, 12)$;
- c. $(0, 0)$ and $(15, 20)$.

Problem 36 (Student page 28) There are many possibilities; these are just examples:

- $(6, 10)$ and $(10, 10)$;
- $(8, 0)$ and $(8, 20)$;
- $(7, 9)$ and $(9, 11)$;
- $(7, 7)$ and $(9, 13)$.

Problem 37 (Student page 28) The coordinates of the midpoint are the averages of the coordinates of the endpoints, and the average of two values is half their sum. So, if $(-2, 1.5)$ is the midpoint, then twice that, $(-4, 3)$, is the pair of coordinates representing the “sum” of the coordinates of the endpoints. If $(-4, 3)$ is the sum, and $(-7, -2)$ is one of the endpoints, then $(3, 5)$ must be the other.

Problem 38 (Student page 28) Parts a and b have many possible answers. Some examples are given here.

- a. $(10, 0)$, $(10, 1)$, $(10, 10)$, $(10, 110)$, $(10, -11)$, $(10, -47)$
- b. $(0, 0)$, $(6, 0)$, $(9, 0)$, $(9, 10)$, $(9, -99)$, $(-10, -100)$
- c. The point must have a value of 10 for the x -coordinate (and therefore lie on the line $x = 10$, the perpendicular bisector of \overline{PQ}).

Problem 39 (Student page 28)

- a. Diagonal \overline{AC} has endpoints $(4, 1)$ and $(7, 1)$ and midpoint $(5.5, 1)$. The midpoint of \overline{BD} is also $(5.5, 1)$. The diagonals bisect each other because they intersect at each other’s midpoint.

- b.** Because the diagonals intersect at each other's midpoint, $ABCD$ must be a parallelogram. The proof that it is a parallelogram uses SAS arguments to show that opposite pairs of triangles in this four-sided figure are congruent, and therefore that the opposite sides of the figure are congruent.

Problem 40 (*Student page 28*) The lines intersect at $(0.5, 0.5)$.

FORMULAS FOR MIDPOINT AND DISTANCE

Problem 1 (*Student page 29*)

- a. $M = (2.5, 7)$
- b. $M = (-2, -0.5)$

Problem 2 (*Student page 30*) Formulas representing the two ideas are presented in sketchy form in the “For Discussion” on page 30 of the Student Module.

Problem 3 (*Student page 30*) $A = (1.5, 1)$ and $B = (35.5, 77.5)$ The two methods produce the same results. Whether they are equally easy is one’s opinion; usually Kesia’s method requires less calculation.

Problem 4 (*Student page 31*)

i	Coordinates of V_i	x_i	y_i
1	(4, 2)	4	2
2	(−2, 4)	−2	4
3	(−4, −2)	−4	−2
4	(2, −4)	2	−4

Problem 5 (*Student page 31*) For $i = 1, i = 2$, and $i = 3$, it is true that $x_i = y_{i+1}$, ($x_1 = y_2$), and so on. But when $i = 4$, the claim says that $x_4 = y_5$, and there is no y_5 , so the claim makes no sense.

Problem 6 (*Student page 31*)

- a. When $i = 2$, the statement is true.
- b. When $i = 4$, the statement doesn’t make sense.

Problem 7 (*Student page 32*) On the square, $y_i = \frac{1}{2}x_i$ only when i is 1 or 3, that is, at vertices V_1 and V_3 .

Problem 8 (*Student page 32*) If $P_1 = (3, 4)$, then, according to the rule, $P_2 = (4, -3)$, $P_3 = (-3, -4)$, and $P_4 = (-4, 3)$.

Problem 9 (*Student page 32*)

- a. The rule says that to derive the new point from the old point, add negative 3 to the x -coordinate of the old point and add 4 to the y -coordinate.

- b. According to the rule, $Q_1 = (1, 6)$, $Q_2 = (-5, 8)$, $Q_3 = (-7, 2)$, and $Q_4 = (-1, 0)$. Plot these points.

Problem 10 (Student page 32) $P_2 = (x_2, y_1)$

Problem 11 (Student page 32) $P_1 = (4, 0)$, $P_2 = (5, 1)$, $P_3 = (6, 2)$, $P_4 = (7, 3)$, $P_5 = (8, 4)$, $P_6 = (9, 5)$, $P_7 = (10, 6)$, and $P_8 = (11, 7)$. All of these points are collinear.

Problem 12 (Student page 32) One example is “To find the coordinates of the midpoint between two points, the x -coordinate is equal to the average of the two x -coordinates and the y -coordinate is equal to the average of the two y -coordinates.”

Problem 13 (Student page 33) Midway between x_1 and x_2 is $\frac{x_1+x_2}{2}$. The segment for which we are finding a midpoint is horizontal, so *all* points on it have the same second coordinate. Therefore, the coordinates of M_1 are $(\frac{x_1+x_2}{2}, y_1)$.

Problem 14 (Student page 33)

- a. If you used $\frac{x_1+x_2}{2}$, then

$$\frac{x_1 + x_2}{2} - x_1 = \frac{x_1 + x_2}{2} - \frac{2x_1}{2} = \frac{x_2 - x_1}{2},$$

which is the same as

$$x_2 - \left(\frac{x_1 + x_2}{2}\right) = \frac{x_2 - x_1}{2}.$$

- b. The algebra above shows that the distance from the midpoint to one endpoint is the same as the distance from the other endpoint to the midpoint.

Problem 15 (Student page 33) The same reasoning shows that the midpoint M_2 of (x_2, y_1) and (x_2, y_2) has coordinates $(x_2, \frac{y_1+y_2}{2})$. Working with a vertical line tells us that the first coordinate does not change. Having the first coordinate not change says that all the points are collinear.

Problem 16 (Student page 33)

$$\begin{aligned}
 y_2 - \frac{y_2 + y_1}{2} &= \frac{2y_2}{2} - \frac{y_2 + y_1}{2} \\
 &= \frac{y_2 - y_1}{2} \\
 \frac{y_2 + y_1}{2} - y_1 &= \frac{y_2 + y_1}{2} - \frac{2y_1}{2} \\
 &= \frac{y_2 - y_1}{2}
 \end{aligned}$$

The calculation $\frac{y_2 + y_1}{2}$ uses Kesia's method of averaging. The algebraic statement

$$y_2 - \frac{y_2 + y_1}{2} = \frac{y_2 + y_1}{2} - y_1$$

uses Paul's logic and says that the distance $y_2 - \frac{y_2 + y_1}{2}$ from the alleged middle, $\frac{y_2 + y_1}{2}$, to one endpoint y_2 must be the same as the distance $\frac{y_2 + y_1}{2} - y_1$ from the other endpoint y_1 to the alleged middle. Paul uses that logic to *find* the middle.

Problem 17 (Student page 34) $M = (329.5, 55.5)$ **Problem 18** (Student page 34) $M = (1888, 31.5)$

Problem 19 (Student page 34) Plotting the three known points shows that $(-1, 5)$ and $(3, -3)$ are the two opposite vertices. So the center of the square must be midway between them, at $(1, 1)$. That must also be the midpoint of the diagonal between the second vertex, $(5, 3)$, and the fourth vertex, so the fourth vertex must be $(-3, -1)$.

Problem 20 (Student page 34) These points are hard to plot, but a rough sketch will help you solve the problem. Such a sketch will show that $(-114, 214)$ and $(186, 114)$ are adjacent vertices of the square, as are $(-114, 214)$ and $(-214, 86)$, while $(-114, 214)$ and $(86, -186)$ are opposite vertices. (This can be verified by calculating the distances between these three pairs of points.) Label the three known vertices $A = (186, 114)$, $B = (-114, 214)$, and $C = (-214, -86)$.

Let D be the fourth vertex of the square. We want to find its coordinates. Because A is 300 units to the right of and 100 units down from B , we see that D will be 300 units to the right of and 100 units below C . Thus, D is the point $(86, -186)$.

The center of the square is the midpoint of both diagonals. \overline{AC} and \overline{BD} are the diagonals of square $ABCD$. Apply the midpoint formula:

$$M_{\overline{AC}} = \left(\frac{186 + (-214)}{2}, \frac{114 + (-86)}{2} \right) = (-14, 14)$$

$$M_{\overline{BD}} = \left(\frac{-114 + 86}{2}, \frac{214 + (-186)}{2} \right) = (-14, 14).$$

Thus, the center of the square is $(-14, 14)$. (We only needed to find one midpoint, but verifying that both diagonals have the same midpoint serves as a check of our work.)

Problem 21 (Student page 34) The midpoint of any diameter is the center of the circle.

a. $(-33, 561)$

b. $(3x, 0.5y)$

Problem 22 (Student page 34) $E = (7, 18)$. See the solution to Problem 37 in Investigation 5.3 if an explanation is needed.

Problem 23 (Student page 35) Since A and B have the same y -coordinate, the line through them is horizontal. The point P will lie on this horizontal line, so its y -coordinate is also 1. To find the x -coordinate of P , add one third the difference between x -coordinates of A and B to the x -coordinate of A :

$$\begin{aligned} \frac{1}{3}(32 - 2) &= \frac{1}{3}(30) = 10 \\ 2 + 10 &= 12. \end{aligned}$$

Therefore, $P = (12, 1)$.

Problem 24 (Student page 35) As in Problem 23, the two given points have the same y -coordinate, so we can use the same method. The y -coordinate of Q is 2. Find the x -coordinate:

$$\begin{aligned} \frac{1}{4}(30 - 2) &= 7 \\ 2 + 7 &= 9. \end{aligned}$$

Therefore $Q = (9, 2)$.

Problem 25 (Student page 35) This problem is similar to Problems 23 and 24, but here we must work with the x - and y -coordinates separately. The x -coordinate of S is

$$5 + \frac{1}{3}(11 - 5) = 5 + 2 = 7,$$

and the y -coordinate is

$$2 + \frac{1}{3}(-1 - 2) = 2 + (-1) = 1.$$

Therefore, $S = (7, 1)$.

Problem 26 (Student page 36) In parts c, d, and f, use the method shown in the solution for Problem 25.

- a. $D = (1, 3)$
- b. $E = (6, -4)$
- c. $F = (3, -1)$
- d. $G = (3, -1)$
- e. $H = (2, -2)$
- f. $J = (3, -1)$

These calculations illustrate the theorem that states that the three medians of any triangle are concurrent at a point $\frac{2}{3}$ of the way from each vertex of the triangle to the opposite side. This theorem will appear again in the solution for Problem 12 of Investigation 5.19.

Problem 27 (Student page 37)

- a. $A = (2, 4\sqrt{6})$
- b. $B = (5, 5\sqrt{3})$
- c. $C = (6, 8)$
- d. $D = (5\sqrt{2}, 5\sqrt{2})$
- e. $E = (10, 0)$
- f. $F = (0, -10)$
- g. $G = (-8, -6)$

Problem 28 (Student page 37)

- a. i.** Look at the figure in the Student Module. We have a right triangle with hypotenuse of length 5 and one leg of length 2. Use the Pythagorean Theorem to find the length of the other leg:

$$a^2 + b^2 = c^2$$

$$2^2 + b^2 = 5^2$$

$$b^2 = 21$$

$$b = \sqrt{21}.$$

Thus, the third vertex of the figure is $(2, \sqrt{21})$.

Although the figure shows the triangle placed in quadrant I, the wording of the problem does not require this. The triangle could also be placed in quadrant IV. In this case, the third vertex would be $(2, -\sqrt{21})$.

- ii.** This is the same as part a(i), except that the hypotenuse now has length 6. For the figure shown in the Student Module, we again use the Pythagorean Theorem:

$$a^2 + b^2 = c^2$$

$$2^2 + b^2 = 6^2$$

$$b^2 = 32$$

$$b = \sqrt{32} = 4\sqrt{2}.$$

For this triangle, the third vertex is $(2, 4\sqrt{2})$. If the triangle is placed in quadrant IV, the third vertex is $(2, -4\sqrt{2})$.

- b.** First, assume that the triangle is placed in quadrant I. Let x be the length of each leg. (This is an isosceles right triangle.) Use the Pythagorean Theorem to find x :

$$x^2 + x^2 = 5^2$$

$$2x^2 = 25$$

$$x^2 = \frac{25}{2}$$

$$x = \sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

Therefore, the vertex on the x -axis is $(\frac{5\sqrt{2}}{2}, 0)$, and the third vertex is $(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2})$.

But that is only one of four possible answers! In part a, the second vertex, $(2, 0)$, is fixed on the positive side of the x -axis, but here, the second vertex

can be on the positive or negative side of the x -axis. It follows that the third vertex can be in any of the four quadrants. Here are all the possibilities for the coordinates of the second and third vertices:

$$\begin{aligned} &\left(\frac{5\sqrt{2}}{2}, 0\right) \text{ and } \left(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right) \\ &\left(\frac{5\sqrt{2}}{2}, 0\right) \text{ and } \left(\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right) \\ &\left(-\frac{5\sqrt{2}}{2}, 0\right) \text{ and } \left(-\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right) \\ &\left(-\frac{5\sqrt{2}}{2}, 0\right) \text{ and } \left(-\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right). \end{aligned}$$

Problem 29 (Student page 37) Let M be the midpoint of \overline{BC} . Then \overline{AM} is the median to \overline{BC} . Using the midpoint and distance formulas, we find that

$$M = (5, 3)$$

and

$$\begin{aligned} AM &= \sqrt{(5-2)^2 + (3-1)^2} \\ &= \sqrt{9+4} \\ &= \sqrt{13}. \end{aligned}$$

Problem 30 (Student page 38)

- a. $\triangle DEF$ is isosceles because $DE = 5\sqrt{5}$ and $EF = 5\sqrt{5}$.
- b. Because it's an isosceles triangle, the altitude from E , the vertex shared by the two congruent sides, to \overline{FD} will also be a median. The midpoint of \overline{FD} is $(7, 2)$, and the length of the segment connecting E to the midpoint of \overline{FD} is 10, so the length of the altitude is 10.

Problem 31 (Student page 38)

- a. $M = (12, 4.5)$, and, since M is the midpoint of \overline{DE} , it is equidistant from D and E ($ME = MD = \frac{5\sqrt{5}}{2}$). Now find ML :

$$\begin{aligned} ML &= \sqrt{(12 - 7)^2 + (4.5 - 2)^2} \\ &= \sqrt{25 + 6.25} \\ &= \sqrt{31.25} = \sqrt{\frac{125}{4}} = \frac{5\sqrt{5}}{2}. \end{aligned}$$

- b. Because \overline{EL} is an altitude to \overline{FD} , by definition \overline{EL} is perpendicular to \overline{FD} , and therefore $\angle ELD$ is a right angle, so $\triangle ELD$ is a right triangle.

Problem 32 (Student page 38) Here the problem is solved for one set of four particular points, $A = (3, 1)$, $B = (4, 7)$, $C = (11, 3)$, and $D = (8, -5)$. The midpoints of the four sides of the quadrilateral are: $M_{\overline{AB}} = (3.5, 4)$, $M_{\overline{BC}} = (7.5, 5)$, $M_{\overline{CD}} = (9.5, -1)$, and $M_{\overline{AD}} = (5.5, -2)$.

$d(M_{\overline{AB}}, M_{\overline{BC}})$ is shorthand for the distance between $M_{\overline{AB}}$ and $M_{\overline{BC}}$.

To show that the shape you get when you connect these midpoints is a parallelogram, show that you have two pairs of congruent opposite sides:

$$d(M_{\overline{AB}}, M_{\overline{BC}}) = d(M_{\overline{AD}}, M_{\overline{CD}}) = \sqrt{17}$$

and

$$d(M_{\overline{AD}}, M_{\overline{AB}}) = d(M_{\overline{BC}}, M_{\overline{CD}}) = 2\sqrt{10}.$$

Problem 33 (Student page 38) $M = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$. Refer to the figure in the Student Module and apply the distance formula:

$$\begin{aligned} MQ &= \sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2} \\ &= \sqrt{\left(\frac{x_1 + x_2 - 2x_1}{2}\right)^2 + \left(\frac{y_1 + y_2 - 2y_1}{2}\right)^2} \\ &= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2} \end{aligned}$$

$$\begin{aligned}
 MS &= \sqrt{\left(x_2 - \frac{x_1 + x_2}{2}\right)^2 + \left(y_1 - \frac{y_1 + y_2}{2}\right)^2} \\
 &= \sqrt{\left(\frac{2x_2 - x_1 - x_2}{2}\right)^2 + \left(\frac{2y_1 - y_1 - y_2}{2}\right)^2} \\
 &= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_1 - y_2}{2}\right)^2} \\
 &= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}.
 \end{aligned}$$

Since M is the midpoint of \overline{RS} , we know that $MS = MR$. We have shown that $MQ = MS = MR$, that is, the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

Problem 34 (Student page 38) Let the vertices of the triangle be $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$. Find the midpoints of two sides:

$$\begin{aligned}
 M_{\overline{AB}} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right); \\
 M_{\overline{BC}} &= \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right).
 \end{aligned}$$

The length of \overline{AC} , the third side of the triangle, is

$$AC = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}.$$

The length of $\overline{M_{\overline{AB}}M_{\overline{BC}}}$ is

$$d(M_{\overline{AB}}, M_{\overline{BC}}) = \sqrt{\left(\frac{x_2 + x_3}{2} - \frac{x_1 + x_2}{2}\right)^2 + \left(\frac{y_2 + y_3}{2} - \frac{y_1 + y_2}{2}\right)^2}.$$

This expression simplifies to

$$d(M_{\overline{AB}}, M_{\overline{BC}}) = \frac{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}}{2} = \frac{1}{2}AC.$$

Problem 35 (Student page 38) To prove the general version of this problem we have to use points such as $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$, and $D = (x_4, y_4)$.

Using this notation, we find the coordinates of the midpoints of the four sides of the quadrilateral:

$$M_{\overline{AB}} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \quad M_{\overline{BC}} = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2} \right)$$

$$M_{\overline{CD}} = \left(\frac{x_3 + x_4}{2}, \frac{y_3 + y_4}{2} \right) \quad M_{\overline{AD}} = \left(\frac{x_4 + x_1}{2}, \frac{y_4 + y_1}{2} \right).$$

Now we have to show that we have two pairs of congruent opposite sides, that is, that $d(M_{\overline{AB}}, M_{\overline{BC}}) = d(M_{\overline{AD}}, M_{\overline{CD}})$, and that $d(M_{\overline{AD}}, M_{\overline{AB}}) = d(M_{\overline{BC}}, M_{\overline{CD}})$.

$$d(M_{\overline{AB}}, M_{\overline{BC}}) = \sqrt{\left(\frac{x_3 - x_1}{2} \right)^2 + \left(\frac{y_3 - y_1}{2} \right)^2} = d(M_{\overline{AD}}, M_{\overline{CD}})$$

and

$$d(M_{\overline{AD}}, M_{\overline{AB}}) = \sqrt{\left(\frac{x_4 - x_2}{2} \right)^2 + \left(\frac{y_4 - y_2}{2} \right)^2} = d(M_{\overline{BC}}, M_{\overline{CD}}),$$

as desired.

COORDINATES IN THREE DIMENSIONS

Problem 1 (Student page 40)

- a. Examples of three other points: $(3, 4, 10)$, $(3, 4, 1,500)$, and $(3, 4, -647)$.
- b. There's an infinite number of such points; they form a line parallel to the z -axis.

Problem 2 (Student page 40) These points form a plane that is parallel to the xy -plane and 3 units above it.

Problem 3 (Student page 40) This set of points is a plane which contains the z -axis and cuts evenly through the first and third quadrants of the xy -plane.

Problem 4 (Student page 40) This set of points is a plane containing the x -axis.

Problem 5 (Student page 40) This set of points is a line containing the origin. This problem asks how Problem 3 can help you find the answer. Problem 3 tells us that the equation $x = y$ describes a plane which contains the z -axis and cuts evenly through the first and third quadrants of the xy -plane. The equation $y = z$ describes a plane which contains the x -axis and cuts evenly through the first and third quadrants of the yz -plane. These two planes intersect in a line.

Problem 6 (Student page 41)

- a. Each point in this set will have a for the x -coordinate but could have any value for the other coordinates. For example, $(a, 2, 18)$, $(a, -4, 107)$, $(a, -73, -1)$, and (a, a, a) all belong to this set of points. The graph would be a plane parallel to the yz -plane.
- b. This is the set of points of the form $(-2, 1, z)$, where z may be any real number. The graph of these points is a line through the point $(-2, 1, 0)$ and parallel to the z -axis.

Problem 7 (Student page 41) In each case, an entire line of points is selected.

- a. In this case, the points lie somewhere on a line, but they could be anywhere.
- b. In this case, the points are either on the horizontal line $(x, 3)$ or the vertical line $(3, y)$.

- c. This is an ambiguous way of writing the coordinates of a point in 3-dimensional space (just as “3” was an ambiguous way of giving a single coordinate of a point on the plane). Here, two of the coordinates are fixed, and only the third can vary, so the set of points forms a line whose points have the form $(3, 4, z)$, where z is any real number.

Problem 8 (Student page 41) In both cases, an entire plane of points is selected.

Problem 11 (Student page 41) The following shapes result: a line, a plane, a cylinder of infinite height, and a square prism of infinite height.

Problem 12 (Student page 42)

- a. $(0, 0, 5)$, $(0, 5, 0)$, $(5, 0, 0)$, $(0, 0, -5)$, $(0, -5, 0)$, $(-5, 0, 0)$. Also, there are quite a few combinations of $(3, 4, 0)$, $(3, 0, 4)$, $(0, 3, 4)$, $(-4, 3, 0)$, $(-4, -3, 0)$, \dots , and various forms of $(2, 0, \sqrt{21})$, $(2, 1, 2\sqrt{5})$, $(3, 1, \sqrt{15})$, \dots , and of course infinitely many more.
- b. The points form a sphere of radius 5.
- c. In three dimensions, these points form a cylindrical surface around the x -axis.
- d. Here, x is not a constant. As one moves farther from the origin along the x -axis, x grows in magnitude, so a point that’s x units from the x -axis gets farther from the axis as it gets farther from the origin. The entire collection of such points (in three dimensions) produces a cone. (Some students interpreted this question to mean concentric cylinders, one for each value of x , but when taken in entirety, these would end up being *space-filling* concentric cylinders.)

What would the answer be in two dimensions?

What might it mean to be “ x units from the x -axis” when x is negative? One might choose to interpret it as the absolute value of x .

Problems 13–15 (Student pages 42–43) These answers will all depend on the shape of your room and where you put the origin. If your room is a rectangular prism, then the angle in Problem 15c is 90° .

Problem 16 (Student page 43)

- a. $A = (4, 0, 3)$ differs from $P = (4, 5, 3)$ only in the middle coordinate. The distance, therefore, is simply the difference between those two coordinates: 5.
- b. $B = (4, 5, 0)$ is 3 units away from $P = (4, 5, 3)$.
- c. $C = (0, 5, 0)$ and $P = (4, 5, 3)$ differ by 3 units in one direction and

by 4 units in a perpendicular direction. The distance between them is $\sqrt{3^2 + 4^2} = 5$.

- d. The distance between $D = (0, 0, 3)$ and $P = (4, 5, 3)$ is $\sqrt{4^2 + 5^2} = \sqrt{41}$.
- e. We can use the Pythagorean Theorem twice to find the distance between $E = (-3, 6, -3)$ and $P = (4, 5, 3)$. The result is the same as computing $\sqrt{7^2 + (-1)^2 + 6^2} = \sqrt{86}$.
- f. The distance from $P = (4, 5, 3)$ to the origin is $\sqrt{50}$.

Problem 17 (Student page 44) Find the differences in their coordinates along each axis, square the results, add them, and take the square root of that sum:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Problem 18 (Student page 44)

- a. The midpoint of \overline{PB} is $(4, 5, 1\frac{1}{2})$.
- b. The midpoint of \overline{PC} is $(2, 5, 1\frac{1}{2})$.
- c. The midpoint of \overline{OP} is $(2, 2\frac{1}{2}, 1\frac{1}{2})$.
- d. The midpoint of \overline{PE} is $(\frac{1}{2}, 5\frac{1}{2}, 0)$.

Problem 19 (Student page 44) To find the coordinates of the midpoint, average the x -, then the y -, and then the z -coordinates of the two points:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

Problem 20 (Student page 44) The farthest vertex of this square has coordinates $(1, 1)$ and is $\sqrt{2}$ units from the origin.

Problem 21 (Student page 44) The farthest vertex of this cube has coordinates $(1, 1, 1)$, and is $\sqrt{3}$ units from the origin.

Problem 22 (Student page 44) It seems reasonable to extend the idea by assigning the farthest vertex of this “hypercube” the coordinates $(1, 1, 1, 1)$. If the same rules apply (and why shouldn’t they?), then that vertex is $\sqrt{4} = 2$ units from the origin.

Another method would be to find an intermediate point, make a right triangle, and use the Pythagorean Theorem twice.

SHAPES IN THE PLANE AND IN SPACE

Problem 1 (*Student page 45*) Some points that produce 5 are $(0, \sqrt{5})$, $(-\sqrt{5}, 0)$, $(1, 2)$, $(2, 1)$, $(-2, -1)$, $(-2, 1)$, $(1, -2)$, and $(\frac{1}{2}\sqrt{19}, \frac{1}{2})$. If you look at all the points that produce a result of 5, you get a circle of radius $\sqrt{5}$. You also get circles if you look at all the points that produce 13, 625, 100, 169, and 1. Every circle contains infinitely many points; the points listed here are just examples.

- a. $(0, \sqrt{13})$, $(\sqrt{13}, 0)$, $(3, 2)$, $(3, -2)$, $(-3, 2)$, $(-3, -2)$, $(2, 3)$, $(1, 2\sqrt{3})$
- b. $(0, 25)$, $(15, 20)$, $(-20, 15)$, $(5, 10\sqrt{6})$, $(5\sqrt{21}, 10)$, $(9, 4\sqrt{34})$, $(17, 4\sqrt{21})$, $(18, \sqrt{301})$
- c. $(0, 10)$, $(0, -10)$, $(-10, 0)$, $(6, 8)$, $(-8, -6)$, $(2, 4\sqrt{6})$, $(2\sqrt{21}, 4)$, $(-5, 5\sqrt{3})$
- d. $(0, 13)$, $(0, -13)$, $(12, 5)$, $(12, -5)$, $(-12, -5)$, $(-12, 5)$, $(8, \sqrt{105})$, $(3, 4\sqrt{10})$
- e. $(0, 1)$, $(0, -1)$, $(0.6, 0.8)$, $(0.6, -0.8)$, $(-0.8, 0.6)$, $(0.2, 0.4\sqrt{6})$, $(-0.2, -0.4\sqrt{6})$, $(0.5\sqrt{2}, 0.5\sqrt{2})$

Problem 2 (*Student page 45*)

- a. You still get a circle, this one of radius 13, centered at the point $(4, -1)$ rather than at the origin.
- b. Eight such points are $(4, -14)$, $(4, 12)$, $(17, -1)$, $(-9, -1)$, $(16, 4)$, $(-8, 4)$, $(-8, -6)$, are $(16, -6)$. Some of these points may be easier to see because a circle 13 units away from the origin would include the various coordinate pairs involving 13 and 0, and 12 and 5. With these all you have to do is add 4 to the x -coordinate and -1 to the y -coordinate, as above. Others need just a bit more calculation, such as $(12, \sqrt{105} - 1)$, and $(7, 4\sqrt{10} - 1)$. In general, to tell if (a, b) is on the figure, check to see if $\sqrt{(4 - a)^2 + (-1 - b)^2} = 13$.

Problem 3 (*Student page 46*)

- a. Eight such points are $(0, 0, -9)$, $(4, 4, 7)$, $(-3, -6, -6)$, $(-3, 6\sqrt{2}, 0)$, $(-3, -5, \sqrt{47})$, $(3, 4, 2\sqrt{14})$, $(1, 2, \sqrt{76})$, and $(5, 6, \sqrt{20})$. All the points that produce 81 with that recipe make a sphere of radius 9.
- b. All the points that are 7 units from the origin make a sphere of radius 7, centered at the origin. Some of these points are $(-2, -3, -6)$, $(2\sqrt{2}, 4, 5)$, and $(0, 0, 7)$.

Problem 4 (Student page 46)

- a. $10^2 + 0 + 0 = 100$
- b. $0 + (-10)^2 + 0 = 100$
- c. $36 + 64 + 48 = 148$
- d. $36 + 64 - 48 = 52$
- e. $25 + 75 - 25\sqrt{3} = 100 - 25\sqrt{3}$
- f. $25 + 75 + 25\sqrt{3} = 100 + 25\sqrt{3}$

Problem 5 (Student page 46) Six such points are $(0, 1)$, $(0, -1)$, $(1, 0)$, $(-1, 0)$, $(1, -1)$, and $(-1, 1)$. Some others, which are rough approximations, are $(0.6, 0.55)$ and $(0.8, 0.32)$. You get an ellipse if you draw the picture of all the points.

Problem 6 (Student page 47) Some points are $(1, 1)$, $(-1, -1)$, $(-1, 1)$, $(1, -1)$, $(0, 2)$, $(0, -2)$, $(\frac{2}{3}\sqrt{3}, 0)$, and $(-\frac{2}{3}\sqrt{3}, 0)$. This is also an ellipse.

Problem 7 (Student page 47)

- a. When the sum of the coordinates is 12, $x + y = 12$, and the shape is a straight line through the points $(0, 12)$ and $(12, 0)$.
- b. When the difference of the coordinates is 12, $x - y = 12$ or $y - x = 12$. The shape is again a straight line, either through the points $(0, -12)$ and $(12, 0)$ or through $(0, 12)$ and $(-12, 0)$.
- c. When the product of the coordinates is 12, $xy = 12$. Points like $(\frac{1}{10}, 120)$, $(\frac{1}{2}, 24)$, $(1, 12)$, $(2, 6)$, $(3, 4)$, $(4, 3)$, $(6, 2)$, and so on are on the graph. So are the points $(-\frac{1}{10}, -120)$, $(-\frac{1}{2}, -24)$, $(-1, -12)$, $(-2, -6)$, and so on. The shape is a hyperbola.
- d. When the ratio of the coordinates is 12, $\frac{x}{y} = 12$ or $\frac{y}{x} = 12$. Each of these sets of points produces a line. $\frac{y}{x} = 12$ is equivalent to saying $y = 12x$. That line includes points like $(1, 12)$, $(2, 24)$, $(3, 36)$, and so on.

Problem 8 (Student page 47) Here are some ideas for recipes:

- $y - 2x = 0$
- $y - 2x = 2$
- $y - x^2 = 2$
- $yx = 0$.

Problem 9 (Student page 47) We know that the third vertex has to be directly over the midpoint of the base, so we have two possibilities: $(5, y)$ and $(5, -y)$. We'll solve for the first one. Because the triangle is equilateral, all its sides are congruent and have length 8. We have

$$\begin{aligned} 8 &= \sqrt{(5-1)^2 + (y-0)^2} \\ 64 &= 16 + y^2 \\ 48 &= y^2, \end{aligned}$$

so $y = 4\sqrt{3}$. The coordinates for the third vertex of the equilateral triangle are $(5, 4\sqrt{3})$ or $(5, -4\sqrt{3})$.

Or, perhaps you remembered that the altitude of an equilateral triangle will form two 30–60–90 triangles whose sidelengths have ratios of 1, $\sqrt{3}$, 2. Thus, the altitude of this equilateral triangle has a length of $4\sqrt{3}$ units.

Problem 10 (Student page 47)

$$\begin{aligned} AB &= \sqrt{34} = A'B' \\ BC &= \sqrt{29} = B'C' \\ CA &= \sqrt{13} = C'A' \end{aligned}$$

By SSS, $\triangle ABC \cong \triangle A'B'C'$.

Problem 11 (Student page 47) $AB = \sqrt{960^2 + 512^2}$ and $A'B' = \sqrt{3840^2 + 2048^2}$, so

$$AB = \frac{1}{4}A'B' \text{ because } \sqrt{3840^2 + 2048^2} = \sqrt{(4 \cdot 960)^2 + (4 \cdot 512)^2}.$$

$$BC = \sqrt{897^2 + 496^2} \text{ and } B'C' = \sqrt{3588^2 + 1984^2}, \text{ so}$$

$$BC = \frac{1}{4}B'C' \text{ because } \sqrt{3588^2 + 1984^2} = \sqrt{(4 \cdot 897)^2 + (4 \cdot 496)^2}.$$

$$CA = \sqrt{63^2 + 16^2} \text{ and } C'A' = \sqrt{252^2 + 64^2}, \text{ so}$$

$$CA = \frac{1}{4}C'A' \text{ because } \sqrt{252^2 + 64^2} = \sqrt{(4 \cdot 63)^2 + (4 \cdot 16)^2}.$$

The corresponding sides of $\triangle ABC$ are proportional to the sides of $\triangle A'B'C'$. By SSS for similarity, $\triangle ABC \sim \triangle A'B'C'$.

Problem 12 (Student page 48) There are other tests for congruence, but SSS is the easiest one to use when you're just looking at the coordinates of the vertices. For each triangle, find the distance between each of the three vertices and then compare

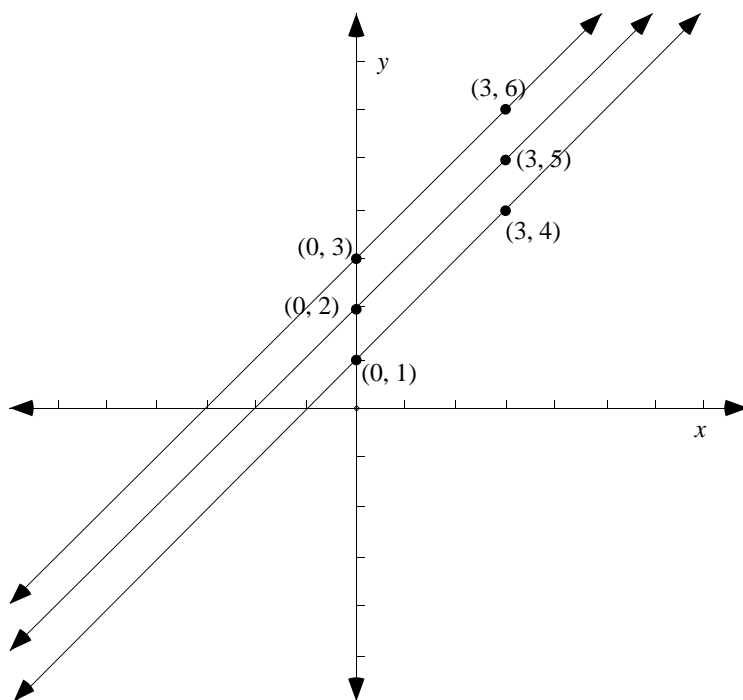
corresponding sides. The two triangles are congruent when the corresponding sides are congruent.

Problem 13 (Student page 48) It's the same idea here; you use the coordinates of the vertices to calculate the sidelengths and compare corresponding sides. If the sides of one triangle are proportional to the sides of the other triangle, then by SSS for similarity, the two triangles are similar.

Problem 14 (Student page 48) Some points are $(-18, -15)$, $(0, 3)$, $(1, 4)$, $(4.33, 7.33)$, $(3, 6)$, and $(7, 10)$. The regularity to the points is that they all lie on the same line—they're collinear. Note that this line makes the same angle with the x -axis as the line in Problem 7 of Investigation 5.3 (although it has moved up the y -axis exactly 3 units).

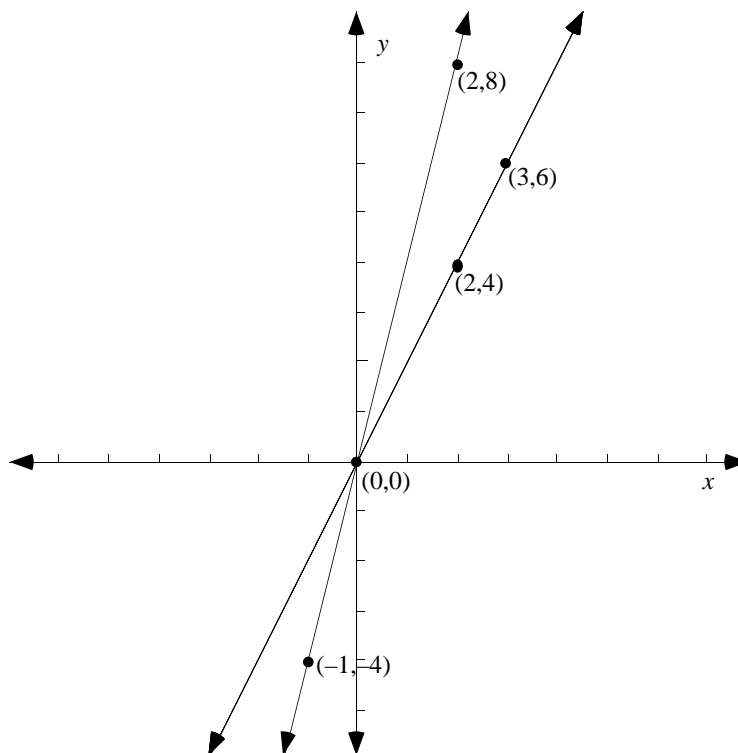
Problem 15 (Student page 48) All the points form a line. Examples: $(0, 2)$, $(3, 5)$, $(3.1, 5.1)$, $(-8, -6)$ and $(-2, 0)$.

Problem 16 (Student page 48) All the points form a line. Examples: $(0, 1)$, $(3.9, 4.9)$, $(3, 4)$ and $(-8, -7)$.



The lines for Problems 14, 15, and 16

Problem 17 (Student page 48) Each of these sets of points also forms a line.



The lines for parts a and c

Problem 18 (Student page 49) All the lines created by points of the form $(x, x + \text{something})$ are parallel. They all have the same slope: because we're using a Cartesian plane in which the x - and y -axis have the same scale, these lines form the same angle with the x -axis (45°). No matter what the *something* is in the equation $y = x + \text{something}$, if you move the x -value of any point one unit over to the right, the corresponding y -value will move up one unit.

Problem 19 (Student page 49) The lines created by points of the form $(x, x \times \text{something})$ have the origin as a common point. If this is not clear, draw the picture of the lines whose points have the properties $(x, x \times \frac{1}{3})$, $(x, x \times \frac{1}{2})$, and $(x, x \times 20)$, for example.

Problem 20 (Student page 49) Two points collinear with $(5, 1)$ and $(8, -3)$ are $(2, 5)$ and $(-1, 9)$. You might first sketch the line through these two points on graph paper and look for another point on the line. Then develop instructions such as those in the next problem.

Problem 21 (Student page 49) One way to do it is to start at a point on the line and move 3 units directly to the right, and then directly down 4 units. When you do this from $A = (5, 1)$, you get $B = (8, -3)$. So continuing from B , you can get $C = (11, -7)$, $D = (14, -11)$, $E = (17, -15)$, and so on. This method works because, for any three points on the same line, the ratios of the distances between the y -coordinates to the distances between the x -coordinates (that is, the slopes) will always be equal. This is true because the triangles formed by those distances and the line in question are similar triangles.

If $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then point $T = (a_1, b_2)$ forms a right triangle with A and B , and it's similar to the triangle formed by $A = (a_1, a_2)$, $C = (c_1, c_2)$, and $U = (a_1, c_2)$. Thus, using $A = (5, 1)$ and $B = (8, -3)$, for example, the slope of \overrightarrow{AB} is $-\frac{3}{4}$, and the slope of \overrightarrow{AC} is $-\frac{6}{8} = -\frac{3}{4}$.

Problem 22 (Student page 49) The slope of \overrightarrow{PR} is

$$\frac{-30 - (-14)}{-40 - (-4)} = \frac{-16}{-36} = \frac{4}{9},$$

while the slope of \overrightarrow{RS} is

$$\frac{20 - (-30)}{80 - (-40)} = \frac{50}{120} = \frac{5}{12}.$$

The slope of a line is constant, so P , R , and S cannot lie on the same line. Thus, P is not collinear with R and S .

Problem 23 (Student page 49) From Problem 22, the slope of \overrightarrow{RS} is $\frac{5}{12}$.

The slope of a line is constant (or invariant): You will get the same value using *any* two points on the line. Two points collinear with R and S are $A = (68, 15)$ and $B = (92, 25)$. Check the slopes:

The slope of \overrightarrow{AS} is

$$\frac{20 - 15}{80 - 68} = \frac{5}{12},$$

and the slope of \overrightarrow{BR} is

$$\frac{25 - (-30)}{92 - (-40)} = \frac{55}{132} = \frac{5}{12}.$$

We see that A , B , R , and S are all collinear.

Note: Two (different) lines with the same slope are parallel. Here, we know that we have one line, not two parallel lines, because \overleftrightarrow{RS} and \overleftrightarrow{AS} , for example, have a common point (S). Parallel lines have no points in common.

Problem 24 (Student page 50) Now R and S are any two points $R = (r_1, r_2)$ and $S = (s_1, s_2)$. Then any third point $P = (x, y)$ will be collinear with R and S if

$$\frac{s_2 - r_2}{s_1 - r_1} = \frac{s_2 - y}{s_1 - x}.$$

Problem 25 (Student page 50)

- a. What's being asked here is to find any points on \mathcal{C} that have a first coordinate of 4. Sketching this on graph paper gives a good idea of approximately where those two points will be (and that there are two points). But to ensure that you've got the exact coordinates, use the distance formula since all points on \mathcal{C} must be exactly 5 units from the origin. Thus $4^2 + y^2 = 25$, and $y = 3$ or -3 . There are two points: $(4, 3)$, and $(4, -3)$.
- b. Now the points we want must have an x -coordinate of 3, and thus (from the part a) the other coordinate must be 4 or -4 . There are two points: $(3, 4)$, and $(3, -4)$.
- c. Points on the horizontal line containing $(8, 0)$ will have a y -coordinate of 0. This line is the x -axis. Circle \mathcal{C} will have two points of intersection with this line, at $(-5, 0)$, and $(5, 0)$.
- d. Here the y -coordinate of any intersection points must be 3, so again we have two points: $(-4, 3)$, and $(4, 3)$.
- e. There are no such points!
- f. The points that are 13 units from $(8, 16)$ form a circle with center $(8, 16)$ and radius 13. Call this circle \mathcal{D} . We want to find any points that are on both circles. This problem may be approached either graphically or algebraically.

Graphical Solution Two circles may intersect in two points, one point (tangent circles), or no points. Using graph paper, graph the two circles very carefully. You will find that the circles intersect in two points. (Since the two points are quite close together, an inaccurate graph may suggest that the circles are tangent or don't intersect at all.) The solutions to the problem are simply the coordinates of the two intersection points.

If your graph is drawn accurately, you will find that one of the intersection points appears to be $(3, 4)$. Using the distance formula or the equations given

in the algebraic solution below, you can confirm that this point is indeed 5 units from the origin and 13 units from (8, 16). The second intersection point appears to have fractional coordinates, so we can only approximate them from the graph: x is a little more than 1, and y is a little less than 5. See the algebraic solution below for the exact coordinates of this point.

Algebraic Solution We need to write equations for the two circles, that is, equations that are satisfied by all the points that lie on the circles. To do this, we use the distance formula and then square both sides of the resulting equations:

\mathcal{C} : Any point on this circle is 5 units from the origin, so if (x, y) is on \mathcal{C} ,

$$\begin{aligned}\sqrt{(x-0)^2 + (y-0)^2} &= 5 \\ \sqrt{x^2 + y^2} &= 5 \\ x^2 + y^2 &= 25.\end{aligned}$$

\mathcal{D} : Any point on this circle is 13 units from (8, 16), so if (x, y) is on \mathcal{D} ,

$$\begin{aligned}\sqrt{(x-8)^2 + (y-16)^2} &= 13 \\ (x-8)^2 + (y-16)^2 &= 169.\end{aligned}$$

We now have a system of two equations:

$$\begin{aligned}(1) \quad & x^2 + y^2 = 25 \\ (2) \quad & (x-8)^2 + (y-16)^2 = 169.\end{aligned}$$

Any solutions of this system will represent the intersection points of the two circles. Although both equations are nonlinear (equations of curves rather than lines), we can use a combination of two methods that are commonly used to solve linear systems: the addition (or elimination) method and the substitution method.

First, expand the left side of equation (2) and simplify the result:

$$\begin{aligned}x^2 - 16x + 64 + y^2 - 32y + 256 &= 169 \\ (3) \quad x^2 - 16x + y^2 - 32y &= -151.\end{aligned}$$

Now, multiply equation (1) by -1 and add the result to equation (3):

$$\begin{array}{r} x^2 - 16x + y^2 - 32y = -151 \\ -x^2 \qquad - y^2 \qquad = -25 \\ \hline -16x \qquad - 32y = -176 \end{array}$$

$$(4) \qquad \qquad \qquad x + 2y = 11.$$

Solve equation (4) for x .

$$(5) \qquad \qquad \qquad x = 11 - 2y.$$

Substitute this expression for x in equation (1).

$$(11 - 2y)^2 + y^2 = 25$$

Simplify the left-hand side, and solve the resulting quadratic equation by factoring:

$$\begin{aligned} 121 - 44y + 4y^2 + y^2 &= 25 \\ 5y^2 - 44y + 96 &= 0 \\ (y - 4)(5y - 24) &= 0 \\ y = 4 \quad \text{or} \quad y &= \frac{24}{5} \end{aligned}$$

Finally, substitute each value for y into equation (5) to find the corresponding value for x .

If $y = 4$,

$$x = 11 - 2(4) = 3.$$

If $y = \frac{24}{5}$,

$$x = 11 - 2\left(\frac{24}{5}\right) = \frac{55}{5} - \frac{48}{5} = \frac{7}{5}.$$

We have found the two points that are on both circles: $(3, 4)$ and $(\frac{7}{5}, \frac{24}{5})$.

Problem 26 (Student page 50) Because there are two possible answers for the equilateral triangle described in Problem 9, there will be four possible locations for the fourth vertex of a tetrahedron, two possibilities for each of the bases. Here we will consider the equilateral triangle with vertices $A = (1, 0, 0)$, $B = (9, 0, 0)$, and $C = (5, 4\sqrt{3}, 0)$. To build a regular tetrahedron using this triangle as a base, you must find the coordinates of a point, D , that is located some distance over the center of this base.

Two methods are described here, but there are other ways of finding these coordinates.

Method 1 An equilateral triangle's many "centers" are all located at the same spot—at the intersection of the three altitudes (or three medians, or three angle bisectors). For this, see Method 2. We can also find this point by averaging the three vertices of the triangle:

$$\left(\frac{1 + 9 + 5}{3}, \frac{0 + 0 + 4\sqrt{3}}{3} \right) = \left(5, \frac{4\sqrt{3}}{3} \right).$$

This gives the x - and y -coordinates of the fourth vertex, D . To find D 's z -coordinate, use the first two coordinates to figure out the lengths of two sides of a right triangle. Use the Pythagorean Theorem to find the third length and, from there, the third coordinate of D .

Method 2 The second method involves using the distance formula three times. You know that D is located 8 units from A , B , and C , so three applications of the distance formula give

$$\begin{aligned} 8 &= \sqrt{(x - 1)^2 + (y - 0)^2 + (z - 0)^2} \\ 64 &= (x - 1)^2 + y^2 + z^2; \\ 8 &= \sqrt{(x - 9)^2 + (y - 0)^2 + (z - 0)^2} \\ 64 &= (x - 9)^2 + y^2 + z^2; \\ 8 &= \sqrt{(x - 5)^2 + (y - 4\sqrt{3})^2 + (z - 0)^2} \\ 64 &= (x - 5)^2 + (y - 4\sqrt{3})^2 + z^2. \end{aligned}$$

The algebra might get messy, but we can solve three equations in three variables to get $(x, y, z) = (5, \frac{4\sqrt{3}}{3}, \frac{8\sqrt{6}}{3})$.

INTRODUCTION TO COORDINATES AND VECTORS

Problem 1 (*Student page 51*) The set of all points whose y -coordinate is 4 forms a line parallel to and 4 units above the x -axis.

Problem 2 (*Student page 51*) The points whose y -coordinate is -2 form a line that is parallel to and 2 units below the x -axis (a horizontal line). The points whose x -coordinate is 5 form a line that is parallel to and 5 units to the right of the y -axis (a vertical line).

Problem 3 (*Student page 52*) Every point on the line will have an x -coordinate of -2 ; the line is parallel to the y -axis and includes the points $(-2, -50)$, $(-2, -10)$, $(-2, -1)$, $(-2, 0)$, $(-2, 5)$, and $(-2, 110)$, among others.

Problem 4 (*Student page 52*) The set of points whose first coordinates are the same as their second coordinates form a line through the points $(7, 7)$ and $(-2, -2)$ or (a, a) . Because we are working with perpendicular x - and y -axes that have a scale of 1-to-1, this line will form a 45° with both axes in the first and third quadrants.

Problem 5 (*Student page 52*) Here we have a line through $(-a, a)$ and $(b, -b)$; some points are $(-5, 5)$, $(-1, 1)$, $(3, -3)$, and $(99, -99)$. This line may be obtained by starting with the line in Problem 4 and reflecting it over the y -axis.

Problem 6 (*Student page 52*) All points 5 units from the origin lie on a circle of radius 5 centered at the origin. Some such points are $(0, 5)$, $(0, -5)$, $(5, 0)$, $(-5, 0)$, $(3, 4)$, $(4, 3)$, $(-3, 4)$, $(-4, 3)$, $(3, -4)$, $(4, -3)$, $(-3, -4)$, $(-4, -3)$, and $(2, \sqrt{21})$.

Problem 7 (*Student page 52*) P is 13 units from the origin because $d = \sqrt{(12 - 0)^2 + (-5 - 0)^2} = \sqrt{169} = 13$.

P is 5 units from $(15, -9)$, because $d = \sqrt{(15 - 12)^2 + [-9 - (-5)]^2} = \sqrt{25} = 5$.

Problem 8 (*Student page 52*) To find the distance between 2 points:

Step 1. Subtract the x - and y -coordinates of the first point from the x - and y -coordinates of the second point.

Step 2. Square the two results and then add them.

Step 3. Take the square root of their sum.

Problem 9 (Student page 52)

- a. $M_{\overline{AO}} = (4, -6)$
- b. $E = (-21, 15)$

Problem 10 (Student page 52)

- a. To find the midpoint, whose coordinates we'll name (x_m, y_m) , first add the two x -coordinates of the endpoints and divide the sum by two; this gives x_m . Then find y_m by doing the same: add the two y -coordinates of the endpoints and divide the sum by two.
- b. If you know the midpoint and one endpoint, find the other endpoint by “undoing” the operations in part a. First multiply the x -coordinate of the midpoint by 2, and subtract from that product the x -coordinate of the known endpoint to obtain the x -coordinate of the unknown endpoint. Do the same with the y -coordinates.

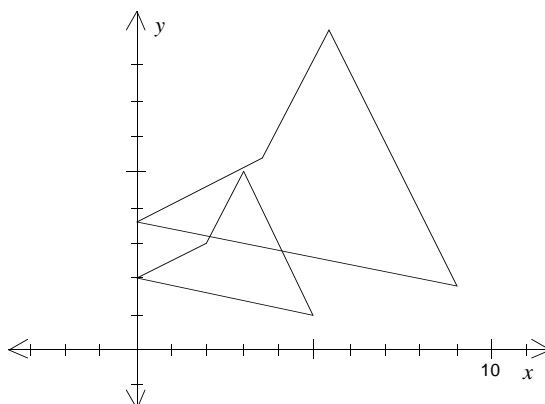
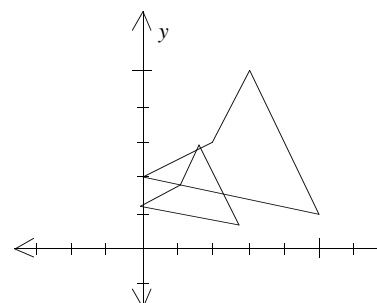
STRETCHING AND SHRINKING THINGS

Problem 1 (Student page 53)

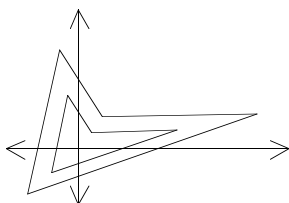
- a.** The two pentagons are similar; the smaller one is scaled down by 2 (or, scaled by $\frac{1}{2}$). The sidelengths of the new pentagon are $\frac{1}{2}$ the sidelengths of the original. The area of the smaller pentagon, however, is only $\frac{1}{4}$ that of the original.
- b.** The coordinates of the vertices of the smaller pentagon are $\frac{1}{2}$ the value of the coordinates of the original pentagon's vertices.
- c.** The coordinates of the vertices of the pentagon scaled up by 2 are (16, 10), (20, 6), (24, 12), (24, 18), and (18, 18).
- d.** The coordinates of the vertices of the pentagon scaled by $\frac{1}{3}$ are $(\frac{8}{3}, \frac{5}{3})$, $(\frac{10}{3}, 1)$, (4, 2), (4, 3), and (3, 3).

Problem 2 (Student page 53)

- a.** The second quadrilateral is a similar copy of the first; its sidelengths are 4 times as long as the original's. The coordinates of the new quadrilateral are (0, 8), (8, 12), (12, 20), and (20, 4).
- b.** If you multiply by -1 , you get a congruent quadrilateral whose vertices are the same distance from the origin as those of the original. The new quadrilateral appears reflected over one axis and then over the other (some people call this "reflecting over the origin"). If you multiply by -2 , the new quadrilateral is a similar copy whose sidelengths are twice as long as the original's.
- c.** In this case, the new quadrilateral is no longer a similar copy. It's distorted because it has grown in the x -direction by a factor of 2, but in the y -direction by a factor of 3.
- d.** A new quadrilateral created in this way would never sit entirely *inside of* the original; (when you create a smaller new quadrilateral, the two lowest coordinates move down closer toward the x -axis). You might say the new quadrilateral contains the original when the multiple is the number 1 because the original and the new quadrilateral would be congruent. But as soon as you make the new quadrilateral larger than the original, it's no longer in a position to contain the original.

scale factor > 1 scale factor < 1

It is certainly possible to start with a different quadrilateral and scale it so that the new quadrilateral is entirely contained by (or entirely contains) the original. Here's one example:



Can you think of others?

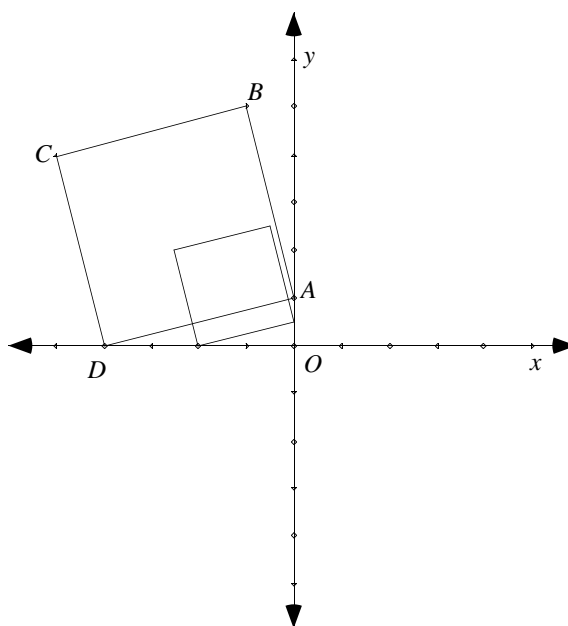
The coordinates of the point $(0, 2)$ change such that its image is just above (for scale factor > 1) or below (for scale factor < 1) the point $(0, 2)$. (Also, multiplying by a negative number results in the entire quadrilateral being placed in a different quadrant and not overlapping the original at all.) Thus, the new quadrilateral will never contain the original.

Otherwise, the new quadrilateral will be disjoint from the original for any multiple, c , such that $c > 2.8$ or $c < \frac{5}{14}$. This can be seen when you choose a multiplier such that $\frac{5}{14} \leq c \leq 2.8$. At 2.8, the line segment between the image of the first point and the image of the last— $(0, 5.6)$ and $(14, 2.8)$ —intersects the original quadrilateral at the third point, $(3, 5)$. For any $c \geq 2.8$, the line segment is raised above the third point, and thus the new quadrilateral is disjoint from the original.

You might find the multiplier of 2.8 by an educated guess-and-check method. This is much harder to do for $\frac{5}{14}$. Here it's useful to know about the equation of the line through the two points $(0, 2)$ and $(5, 1)$. For the quadrilateral and its image to be disjoint, you need to find c so that $(3c, 5c)$ will be below that line. The equation of the line through $(0, 2)$ and $(5, 1)$ is $y = -\frac{1}{5}x + 2$. Thus, we need to have $5c < -\frac{1}{5}(3c) + 2$. This results in what we wanted: $c < \frac{5}{14}$.

Problem 3 (Student page 54)

- a. There are two choices for the locations of C and D : either $C = (3, 6)$ and $D = (4, 2)$, or $C = (-5, 4)$ and $D = (-4, 0)$.
- b. You get a smaller square; each of its vertices is half as far from the origin as those of the original square. The smaller square is a copy of the original scaled by $\frac{1}{2}$, so its sidelengths are half the lengths of the original.



One location for $ABCD$ and its copy scaled by $\frac{1}{2}$

- c. When you scale by -3 , you get a larger square that has been reflected over the origin and whose sidelengths are 3 times those of the original square. (Its area, however, is 9 times greater than that of the original.)

Problem 4 (Student page 54) When you multiply both coordinates of each vertex by some number $n > 0$ and then connect the vertices, you get a similar copy (congruent when $n = 1$) whose sidelengths are n times as long as those of the original and whose vertices are n times as far from the origin. When you multiply both coordinates of each vertex by some number $n \leq 0$, you get an enlarged or reduced copy reflected over the origin (or collapsed into the origin when $n = 0$).

Problem 5 (Student page 55)

- a. To scale Trig by 2, multiply both coordinates of several well-placed points by 2, and sketch the rest.
- b. If you scale by a negative number, you will get a scaled copy that is upside-down *and* backward, since the picture of Trig will be reflected over the origin. For example, you could replace each point (x, y) by the point $(-2x, -2y)$ to get a picture of Trig's head in quadrant III, upside down, facing left, and the same size as the one obtained in part a.

In order to obtain a picture that is upside down, but not backward, as the problem requires, we need to reflect the picture over the x -axis only. This requires multiplying x -coordinates from the original picture by a positive number and the y -coordinates by the negative number with the same absolute value. For example, you could replace each point (x, y) by the point $(2x, -2y)$ to get a picture of Trig's head in quadrant IV, upside down, still facing right, and the same size as the one obtained in part a.

Problem 6 (Student page 58) Any lengths in the new picture should be $\frac{1}{2}$ as long as the corresponding lengths in the first picture. The new picture should be only $\frac{1}{2}$ as far from the origin as the original.

Problem 7 (Student page 59) In Problem 5, we scaled Trig by 2 by doubling the coordinates of the points we chose. In Problem 6, we extended the original vectors to twice their length. The coordinates of the new arrowheads are twice the coordinates of the original ones, so both methods produce the same results.

Problem 8 (Student page 59) To scale a point on a figure by a negative number, n , use a ruler or straightedge to extend a vector in the opposite direction but on the same line as the vector from the origin to the point on the figure. The new vector will point in the opposite direction and will be n times as long as the vector from the origin to the point.

CHANGING THE LOCATION OF THINGS

Problem 1 (Student page 60)

- a. The two pentagons are congruent, but the first pentagon has been translated 8 units to the right to obtain new vertices: (16, 5), (18, 3), (20, 6), (20, 9), and (17, 9).
- b. All the y -coordinates are the same, but the x -coordinates are 8 greater than those of the original. The pentagon is 8 units further away from the y -axis.

Problem 2 (Student page 60)

- a. Now the new y -coordinates are 8 greater than the y -coordinates of the original.
- b. The new x -coordinates are 6 greater, and the y -coordinates are 8 greater than the original coordinates. The two pentagons are still congruent.

Problem 3 (Student page 60)

- a. The new pentagon is congruent to the original but has been translated, or moved, 10 units to the right and 7 units up.
- b. The new pentagon is congruent to the original but has been translated 10 units to the right and 7 units down.

Problem 4 (Student page 60) When you add some number n to the first coordinate of every vertex, the figure moves (or is translated) n units to the right if n is positive or n units to the left if n is negative. When you add some other number m to the second coordinate of every vertex, the figure is translated m units up if $m > 0$ or m units down if $m < 0$.

If $n = 0$, there is no horizontal translation. If $m = 0$, there is no vertical translation. If $n = 0$ and $m = 0$, the figure is not translated at all. It stays in its original location.

Problem 5 (Student page 61)

- a. The two figures are congruent; the only difference is their locations. You have to slide $AKLJ$ 8 units to the right and 6 units up to obtain $A'K'L'J'$.
- b. To obtain the coordinates of $A'K'L'J'$ from the coordinates of $AKLJ$, you have to add 8 units to the x -coordinates and add 6 units to the y -coordinates.

Problem 6 (Student page 61) You get a congruent triangle that has been translated 10 units to the right and 6 units up.

Problem 7 (Student page 62) $(x, y) \mapsto (2x, 2y)$

Problem 8 (Student page 62)

- a. The resulting polygon has been translated 8 units to the right and 5 units up. The figures are congruent.
- b. The polygon has been translated 8 units to the left and 5 units up. The figures are congruent.
- c. The resulting enlarged polygon has sidelengths 3 times as long as those of the original, and its vertices are 3 times as far from the origin as those of the original. The figures are similar.
- d. Again the figures are similar, but in this case the resulting copy has been scaled by $\frac{1}{2}$, and its vertices are only $\frac{1}{2}$ as far from the origin as those of the original polygon's.
- e. Here the resulting polygon is congruent to the one above in part d, but it has been translated 7 units to the right and 10 units up.
- f. The resulting polygon is congruent, but has been reflected over the y -axis and translated up 2 units. The vertices of this polygon, therefore, are the same distance from the y -axis as those of the original, but they are 2 units farther away from the x -axis.
- g. Here the resulting polygon is neither congruent nor similar to the original. It's distorted because only the x -coordinates have been multiplied by two, creating a polygon whose dimensions are relatively wider than they were before. The resulting polygon has been stretched in the x -direction and translated up 2 units.

PICTURES FROM RULES, RULES FROM PICTURES

Problem 1 (*Student page 63*)

- a. $(x, y) \mapsto (x + 5, y - 6)$
- b. $(x, y) \mapsto (x + 9, y - 5)$
- c. $(x, y) \mapsto (\frac{1}{2}x, \frac{1}{2}y)$
- d. $(x, y) \mapsto (\frac{1}{2}x + 8, \frac{1}{2}y - 2)$

Problem 2 (*Student page 66*) You have to move (translate) $\triangle DIG$ 5 units to the right and 1 unit down. You add 5 units to the x -coordinates of the vertices of $\triangle DIG$ and subtract 1 from the y -coordinates. This is written $(x, y) \mapsto (x + 5, y - 1)$.

Problem 3 (*Student page 67*)

- a. With Jorge's method, the vertex $(3, 6)$ is sent to $(16.75, 1.5)$. Using Yutaka's method, the same vertex is sent to $(4.75, 1.5)$.
- b. No
- c. Jorge's is $(x, y) \mapsto \frac{1}{4}(x, y) + (16, 0)$, and Yutaka's is $(x, y) \mapsto \frac{1}{4}(x + 16, y)$.

Problem 4 (*Student page 68*) A to B is $(x, y) \mapsto 2(x, y) + (7, 0)$; A to C is $(x, y) \mapsto 2(x + 7, y)$.

Problem 5 (*Student page 68*)

- a. This is a square, translated 8 units to the right and up 5 units.
- b. This is a square, translated 3 units to the left.
- c. This is a rectangle whose length in the x -direction is 3 times greater, and whose width in the y -direction is 4 times greater than the sidelength of the original square.
- d. This is a square, scaled by -1 . The resulting figure has been reflected over both axes and is located entirely in the third quadrant.
- e. This is a rectangle on the other side of the y -axis from the original square. It is twice as long in the x -direction, the same size in the y -direction and translated down 2 units.
- f. The resulting figure is a square that has been scaled by 2, then reflected over the x -axis, and translated 1 unit to the right. So the new square is located in the fourth quadrant, and its sides are twice as long as the sides of the original.

- g. This is a rectangle located directly above the original square. It has been stretched by a factor of 2 in the y -direction. To get the y -coordinates, you must translate up 1 unit and then scale by 2; the x -coordinates remain unchanged.
- h. This is a square. Because the original square had a line of symmetry coinciding with the line created by the points whose x -coordinates are the same as the y -coordinates, this rule results in a square that maps onto itself.

Problem 6 (Student page 69)

- a. Any rule that involves coefficients for the x - and y -coordinates, except in the case that the coefficients have the same absolute value, will result in some kind of nonsquare rectangle. No matter what numbers are added to or subtracted from the coordinates, the result is a square as long as the two coordinates have coefficients of the same absolute value. $(x, y) \mapsto (\frac{1}{2}x, \frac{1}{2}(y + 2))$ creates a square.
- b. No, because the multiplication of the two coordinates stretches or shrinks the shape in the same direction of the two dimensions, while adding or subtracting a number just moves the shape up or down, to the right or left. Because the square's adjacent sides are perpendicular like the coordinate axes we've chosen to work with, the square never loses its right angles. Likewise, a rectangle transformed by any of these rules would never change into anything but another rectangle (whether square or not).

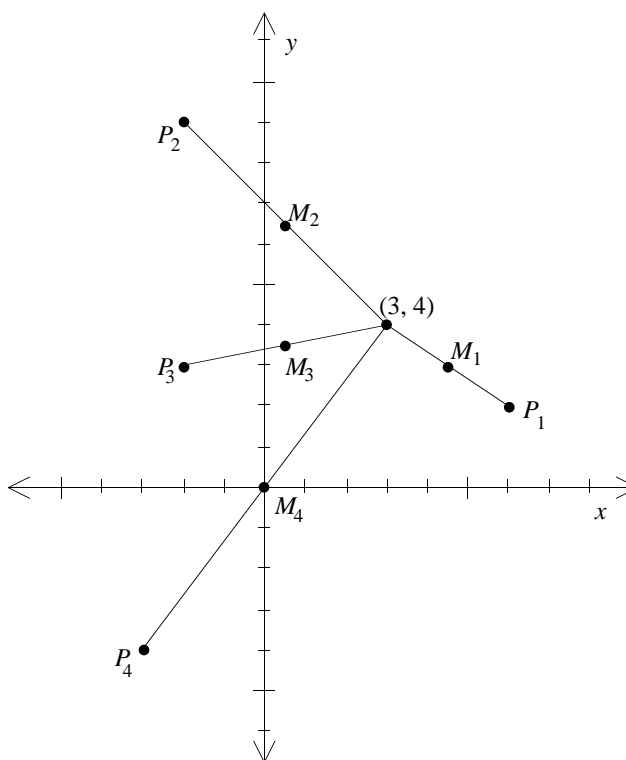
Problem 7 (Student page 69)

- a. Yes, Stella is right.
- b. One way: "Shrink a and b by $\frac{1}{2}$, then move over 3.5 units to the right and 4 units up." This is: $(a, b) \mapsto (\frac{1}{2}a + 3.5, \frac{1}{2}b + 4)$. Alternatively, "Move over 7 to the right and up 8. Then scale by $\frac{1}{2}$." This is: $(a, b) \mapsto (\frac{1}{2}(a+7), \frac{1}{2}(b+8))$.

Problem 8 (Student page 71) First move to the left 4 and up 3. Then stretch (scale) by 2. Then move to the right 6 and down 4. Then shrink (scale) by $\frac{1}{2}$. The result is that all coordinates end up as if they'd been translated to the left 1 unit and up 1 unit.

In the Student Module, an "over and up" was one step, and scaling both coordinates was one step.

Problem 9 (Student page 71) $(a, b) \mapsto (\frac{1}{2}(a + 3), \frac{1}{2}(b + 4))$, or $(a, b) \mapsto \frac{1}{2}(a + 3, b + 4)$



Problem 10 (Student page 71)

- a. This stretches (scales) by 3.
- b. This shrinks by 2 (scales by $\frac{1}{2}$).
- c. This translates 3 in the x -direction, 2 in the y -direction, and negative 1 in the z -direction.
- d. This first translates negative 4 in the x -direction, 3 in the y -direction, and 2 in the z -direction. Then it stretches by 2 and translates 6 in the x -direction, negative 4 in the y -direction, and negative 1 in the z -direction. Finally, it shrinks by 2 (scales by $\frac{1}{2}$).

Problem 11 (Student page 71) The simplified form is $(x, y) \mapsto (x - 1, y + 1)$. We could find it this way:

$$\begin{aligned} &\left(\frac{1}{2}(2(x - 4) + 6), \frac{1}{2}(2(y + 3) - 4)\right) \\ &((x - 4) + 3, (y + 3) - 2) \\ &(x - 1, y + 1). \end{aligned}$$

The second line above is an “intermediate step.”

A point will be sent to the same place whether you apply the rule in Problem 8, or either of these two forms of the rule. Also, because $(2(x + 1), 3(y + 1))$ simplifies to $(2x + 2, 3y + 3)$, the rules for these two expressions are equivalent and will send points to the same places.

Problem 12 (Student page 72) One example of two rules that do different things but have the same end result is

$$(x, y) \mapsto \left(\frac{1}{2}(x - 2), 4(y + 1)\right)$$

and

$$(x, y) \mapsto \left(\frac{1}{2}x - 1, 4y + 4\right).$$

Other examples are the different forms of the rule from Problem 8 that are discussed in Problem 11, as well as the other two rules you’re asked to compare in Problem 11. Algebraically it’s explained by the fact that two mathematical expressions may be equivalent but look different, for example,

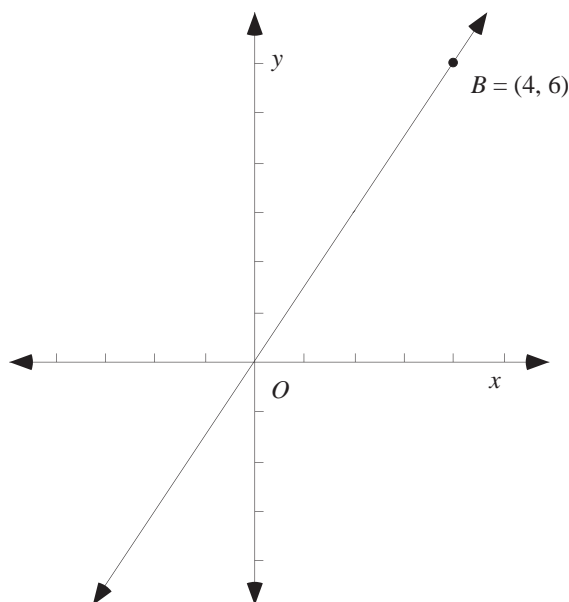
$$\frac{1}{2}[2(x - 4) + 6] = x - 1.$$

A geometric explanation also involves keeping track of the order of operations. If the x -coordinate of a point is first scaled by 2 and then 6 is added to the resulting x -coordinate, the end result will be the same as if you first translated the original x -coordinate by 3, and then scaled the result by 2.

SCALING POINTS

Problem 1 (*Student page 74*) In each part of the problem, A and cA are both on the same line through the origin. Because A is the same point each time in parts c through g, no matter what the value of c , cA seems always to lie on the same line through A and the origin.

Problem 2 (*Student page 74*) This problem strengthens our belief that the graph of every scalar multiple of B will lie on the line through B and the origin. As the value of t ranges over the real numbers, the set of all possible points described by tB creates a line.



Problem 3 (*Student page 74*) As in Problem 1, both A and cA are on the same line through the origin. cA will be c times as far from the origin as A . There is an entire circle of points c times as far from the origin as A ; two of them are on the same line through the origin as A . cA will be the one of those two points that is also in the same direction in relation to the origin as A .

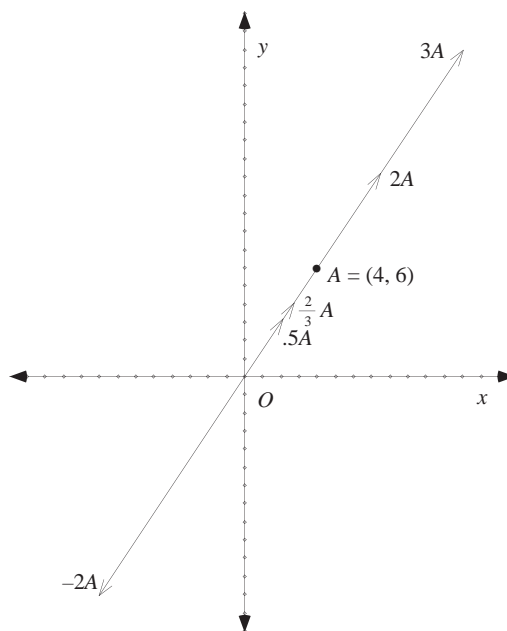
To find $2A$ if you know A , draw \overrightarrow{OA} . Mark off the length OA on the line; then from A mark off that length again. You will be at $2A$. Similarly for $-3A$, mark off the length of A 3 times, but this time mark the length going in the opposite direction as A from the origin.

Algebraically, each coordinate of cA is c times as great as the corresponding coordinate of A .

Problem 4 (Student page 74) When you multiply a number by a number, it may mean that you end up with more or less of something. For example, 4 blocks become 12 blocks when multiplied by 3. But, if you take 4 points and multiply them by 3, you don't get 12 points, but 4 new points whose coordinates are scaled up by 3. Similarly, multiplying 12 blocks by $\frac{1}{2}$ (or, equivalently, dividing by 2) results in 6 blocks, but scaling a group of points by $\frac{1}{2}$ does not result in half as many points.

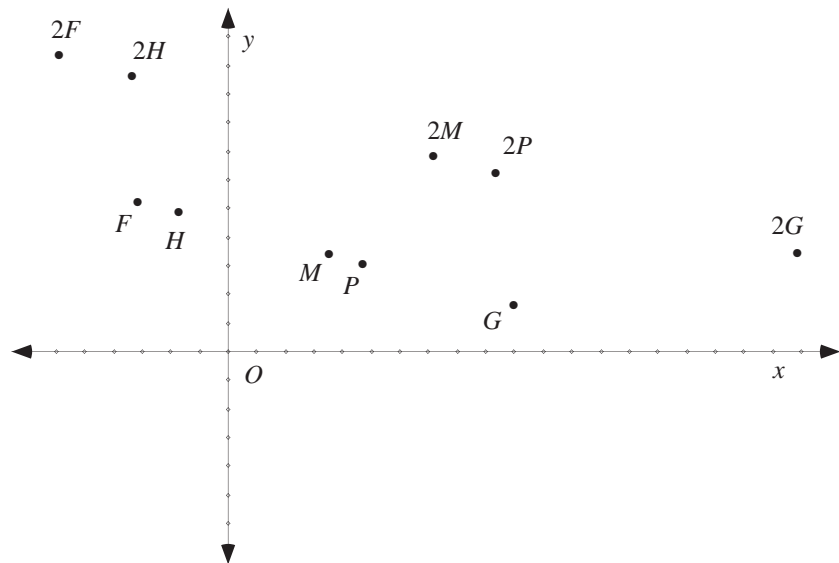
A point is not a number, but a location with regard to a reference point, the origin. If we multiply the coordinates of that location by 3, we get the coordinates of a new location that is three times as far from our reference point.

Problem 5 (Student page 75) The vector from O to A will be $\frac{1}{2}$ as long as the vector from O to $2A$, $\frac{1}{3}$ as long as the vector from O to $3A$, twice as long as the vector from O to $\frac{1}{2}A$, $\frac{3}{2}$ as long as the vector from O to $\frac{2}{3}A$, and half as long as (and reflected over the origin from) the vector from O to $-2A$.

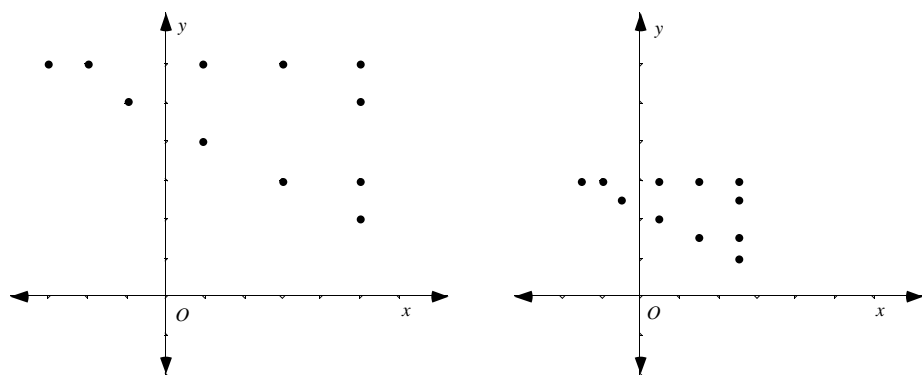


Problem 6 (Student page 76) The points F , H , M , P , and G are all collinear. When scaled by 2, each of these points will be exactly twice the distance from the origin, and will lie on a line parallel to the line containing the first set of points. Also,

the points seem to “spread out” as they are scaled. In fact, the distance between the points under consideration doubles.



Problem 7 (Student page 77) You get a similar picture, but points are half as far from the origin as in the original. This time, distances between the consecutive points are half those in the original.

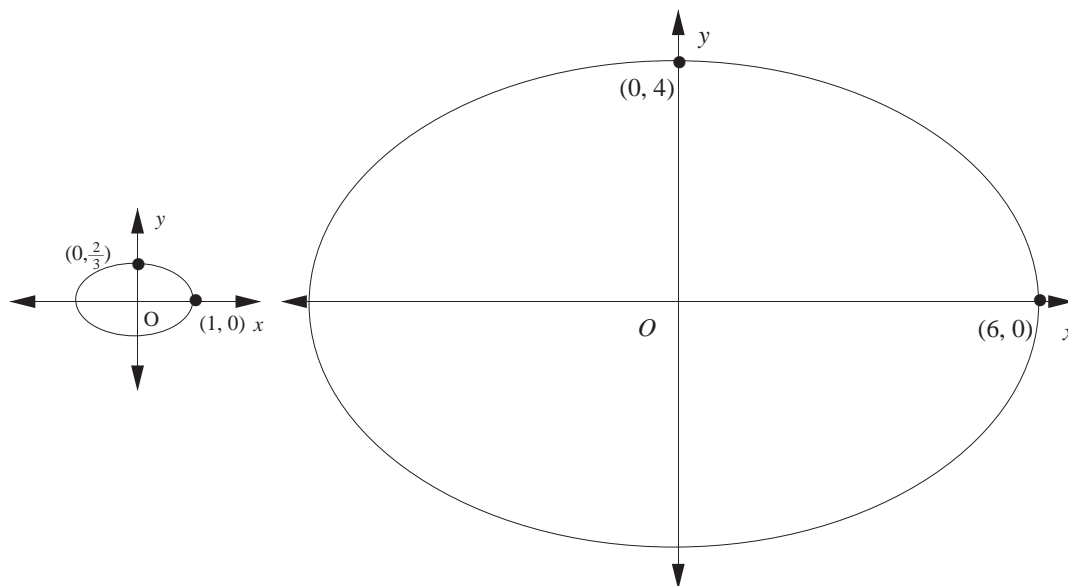


Note that because you are now concerned with the whole circle—a dense set of points—the “gaps” or distances between points don’t get larger because there are no gaps.

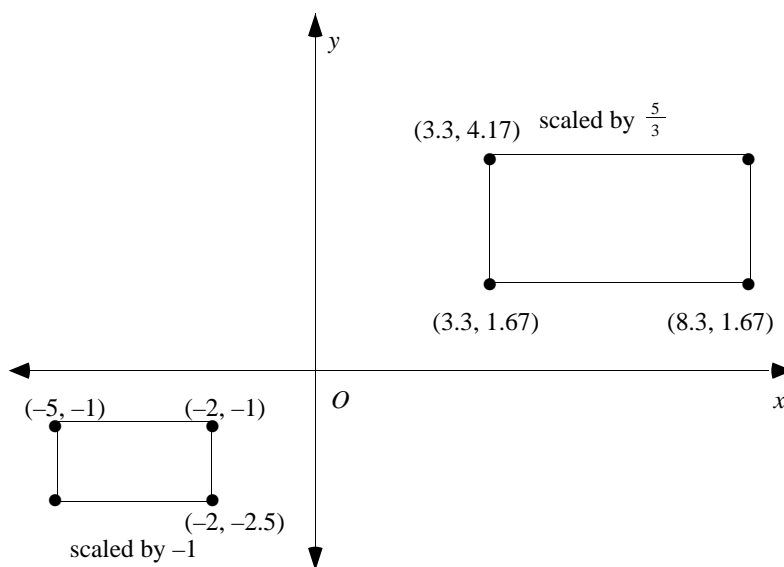
Problem 8 (Student page 77) You get some points on what appears to be a circle centered at the point $(-14, -12)$. The circle’s radius will be 4 times that of the original picture, and because you are only plotting specific points, the distances between them increase as well.

Problem 9 (Student page 78) If you scale by 2, you get a circle centered at the origin of radius 12. If you scale by $\frac{1}{2}$, you get a circle of radius 3. Scaling by $-\frac{1}{2}$ still results in a circle of radius 3 centered at the origin, but the points in each quadrant have been reflected over the origin as well as scaled by $\frac{1}{2}$, so a point originally in the first quadrant gets sent to the third quadrant, and a point originally in the fourth quadrant gets sent to the second quadrant.

Problem 10 (Student page 78) The ellipse with points $(3, 0)$, $(-3, 0)$, $(0, 2)$, and $(0, -2)$, when scaled by $\frac{1}{3}$, shrinks to a similar ellipse with points $(1, 0)$, $(-1, 0)$, $(0, \frac{2}{3})$, and $(0, -\frac{2}{3})$. When scaled by -2 , the ellipse is enlarged, and points on the ellipse in any given quadrant are reflected over the origin.



Problem 11 (Student page 78) Here's what you get if you scale by -1 and by $\frac{5}{3}$:



Problem 12 (Student page 79)

- a. The second picture is obtained by connecting the midpoints of the triangle in the first picture. This results in four similar triangles, three of them oriented in the same direction as the original triangle. We'll call them "right-side-up."

Each picture is a copy of the preceding picture with segments connecting the midpoints of each new right-side-up triangle. So the third picture is obtained by connecting the midpoints of the three right-side-up triangles created in picture 2. The fourth picture is obtained by doing the same to the 9 right-side-up triangles created in picture 3. To create the fifth picture, you have to connect the midpoints of the 27 right-side-up triangles in picture four.

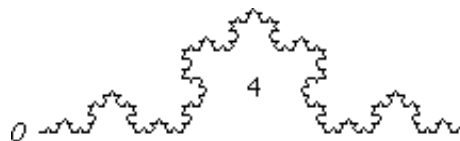
Here is a Logo program that will generate the pictures shown on page 79 of the Student Module. The input n tells the program which picture in the sequence to make.

```
To Sierpinski :n :s
  if :n = 0 [stop]
  repeat 3 [fd :s rt 120
    Sierpinski :n - 1 :s/2]
end
```

- b. If you scale each picture by $\frac{1}{2}$ you end up with a new sequence of pictures, each of which is congruent to one of the right-side-up triangles from the following picture in the original sequence.

Problem 13 (Student page 80)

- a. Each picture is obtained from the one previous to it by dividing every line segment of the previous picture into three equal-length parts, and then erasing the middle part and inserting the other two sides of what would be an equilateral triangle if you hadn't erased the base. Below is the fourth picture in the sequence:



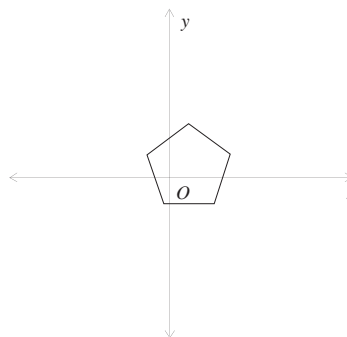
```

To Koch :n :s
  if else :n = 0
    [fd :s]
    [koch :n - 1 :s/3
    lt 60
    koch :n - 1 :s/3
    rt 120
    koch :n - 1 :s/3
    lt 60
    koch :n - 1 :s/3]
  end

```

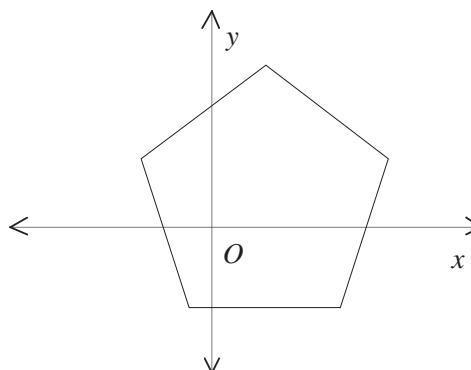
- b. If you scale each picture by one third, you get a copy of a piece of the next picture *before* you try to scale it. So picture 1, after it's scaled, is congruent to the first third of picture 2 before it's scaled.

Problem 14 (Student page 80) Scale this pentagon by 3 to get the desired pentagon (which is the same size as the one shown in the solution for Problem 15).

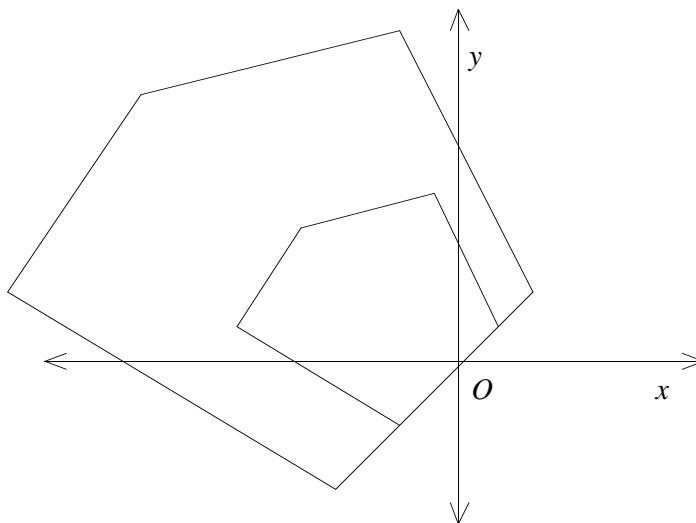


This is the “identity transformation,” like multiplying a real number by 1, or adding 0 to a number.

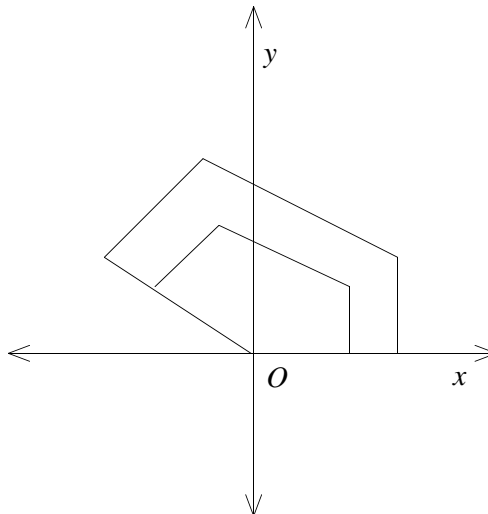
Problem 15 (Student page 81) The pentagon scaled by 1 remains the same; it doesn’t change locations and it doesn’t change size.



Problem 16 (Student page 81) One example looks like this. Note that the essential feature is that one side must contain the origin, but not necessarily as a vertex of that side. (See the next problem.)



Problem 17 (Student page 81) If one side of the pentagon contains the origin as a vertex, then the enlarged scaled copy of the pentagon will have two sides that completely contain the corresponding sides of the original pentagon. Only one point stays in the same spot: the origin. One example looks like this:



Here we are assuming that $A \neq O$.

Problem 18 (Student page 81) Answers will vary. Some possibilities:

- If you scale a point A , the new point cA is on \overleftrightarrow{OA} .
- If $-1 < c < 1$, but $c \neq 0$, cA is closer to the origin than A . If $|c| > 1$, cA is farther from the origin than A . If $c = 0$, $cA = O$.
- If $c < 0$, cA is not in the same quadrant as A . If A is in quadrant I, cA is in quadrant III, and vice versa. If A is in quadrant II, cA is in quadrant IV, and vice versa. So cA is reflected over the origin, which is the same as being reflected over one axis and then the other.

Problem 19 (Student page 81) It appears that $\triangle A$ can be scaled to get $\triangle B$ and vice versa, because the corresponding vertices are collinear with the origin and the triangles appear to be similar. $\triangle A$ appears to be the same size and shape as $\triangle D$, so $\triangle A$ could be translated to produce $\triangle D$ (but because the corresponding points aren't collinear with the origin, scaling could never be the only transformation needed to obtain $\triangle D$ from $\triangle A$).

$\triangle A$ (as well as $\triangle B$) does have vertices that are collinear with the vertices of $\triangle C$ and the origin, but these triangles don't have the same shape. That means, if produced from the vertices of $\triangle A$, each of $\triangle C$'s three vertices has been scaled by a different number.

$\triangle D$ couldn't be scaled to get $\triangle B$ (and vice versa) because the corresponding points aren't collinear with the origin, although one could be scaled and then translated to the other.

Problem 20 (Student page 81) For one figure to be a scaled copy of another, the two figures must be similar. If the two figures are located in the xy -plane, they must be similarly-oriented with respect to the origin, and each point on one figure must be on a line through the origin and the corresponding point on the other figure.

Problem 21 (Student page 81) Someone tall can probably reach $4A$ by standing on a chair, assuming the ceiling is high enough— $4A$ is 8 feet above the floor. If you placed the origin at the corner of the room, however, you probably cannot reach $-1A$, for it will be below the floor and on the other side of the walls. The points A , $2A$, $1.5A$, $\frac{1}{2}A$, and $4A$ will all lie on the same line through the origin you chose.

Problem 22 (Student page 82) The statement is true. Here's a proof: We'll let d_1 = the distance from A to B , and d_2 = the distance from $\frac{1}{2}A$ to $\frac{1}{2}B$.

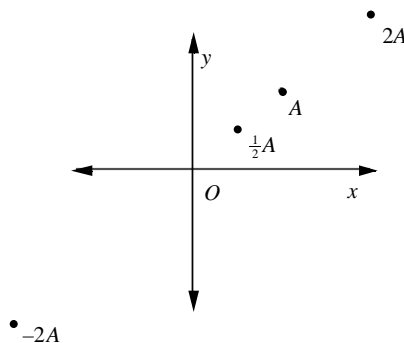
$$d_1 = 10 = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$$

and

$$\begin{aligned}
 d_2 &= \sqrt{\left(\frac{1}{2}b_1 - \frac{1}{2}a_1\right)^2 + \left(\frac{1}{2}b_2 - \frac{1}{2}a_2\right)^2} \\
 &= \sqrt{\left(\frac{1}{2}\right)^2 [(b_1 - a_1)^2 + (b_2 - a_2)^2]} \\
 &= \frac{1}{2}\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2} \\
 &= 5.
 \end{aligned}$$

Problem 23 (Student page 82)

a. Here are the locations:



b. If A is between O and cA , then $c > 1$. If cA is between O and A , then $0 < c < 1$; and if O is between A and cA , then $c < 0$.

Problem 24 (Student page 82)

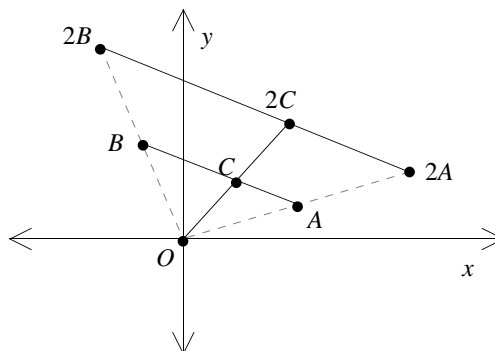
a. True; here's a proof:

$$d(O, A) = \sqrt{(a_1 - 0)^2 + (a_2 - 0)^2} = \sqrt{a_1^2 + a_2^2}$$

and

$$d(O, cA) = \sqrt{(ca_1)^2 + (ca_2)^2} = \sqrt{c^2(a_1^2 + a_2^2)} = |c|\sqrt{a_1^2 + a_2^2}.$$

b. True



By SAS,

$$\triangle OAC \sim \triangle O(2A)(2C)$$

and

$$\triangle OBC \sim \triangle O(2B)(2C).$$

So

$$\angle ACO \cong \angle (2A)(2C)O,$$

and

$$\angle BCO \cong \angle (2B)(2C)O.$$

Also,

$$m\angle ACO + m\angle BCO = 180^\circ;$$

thus,

$$m\angle (2A)(2C)O + m\angle (2B)(2C)O = 180^\circ,$$

and thus $2C$ is on $\overline{(2A)(2B)}$.

c. True; from part b, we know that M is somewhere on the segment from A to B . Now we need only prove the two distances are equal. Let d represent the distance AM . Because M is the midpoint, this means $d = MB$ also. By part a above, the distance from $3A$ to $3M$ equals $3d$, and the distance from $3M$ to $3B$ is also $3d$. These two distances are equal, so M must be the midpoint.

d. False; for simplicity's sake, let B be the origin. Let A and C be any two points that form a 45° angle with B . Then $2A$ and $2C$ will be collinear with A and B , and C and B , respectively. So $\angle (2A)(2B)(2C) = \angle ABC$.

You could also argue this by similarity of triangles. Take any $\triangle ABC$ with a 45° angle and scale it by 2. Scaling it by 2 results in a similar triangle whose angles will all have the same measure, so $\angle (2A)(2B)(2C) = \angle ABC$.

Problem 25 (Student page 82) Calculate the distances OA , $A2A$, and $O2A$:

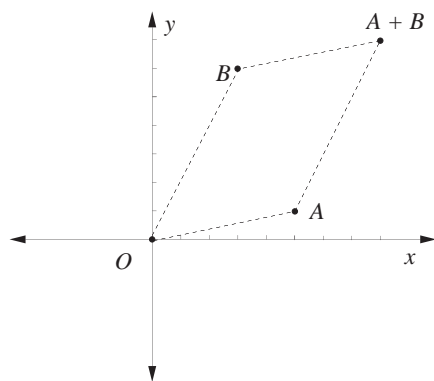
$$\begin{aligned}OA &= \sqrt{5^2 + 12^2} \\&= \sqrt{169} = 13 \\A2A &= \sqrt{(10 - 5)^2 + (24 - 12)^2} \\&= \sqrt{169} = 13 \\O2A &= \sqrt{10^2 + 24^2} \\&= \sqrt{676} = 26\end{aligned}$$

Since the sum of the distances $OA + A2A = O2A$, the points O , A , and $2A$ must be collinear.

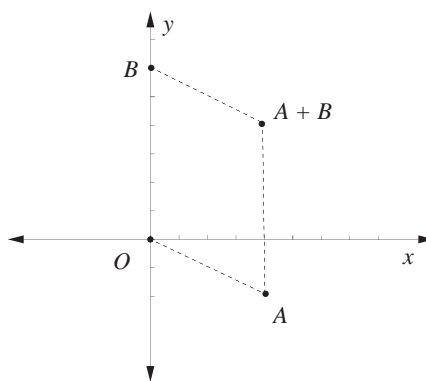
Alternatively, you can use slope: slope $\overline{OA} = \frac{12}{5}$, and slope $\overline{O2A} = \frac{24}{10} = \frac{12}{5}$. Both segments have the same slope and contain the origin, so they must lie on the same line.

ADDING POINTS

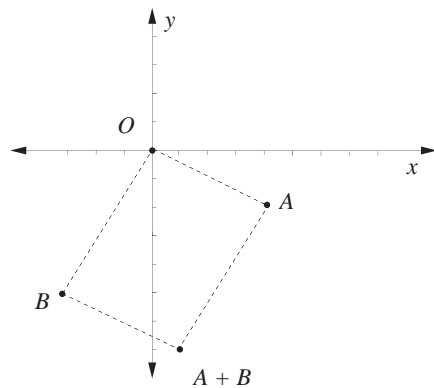
Problem 1 (Student page 85)



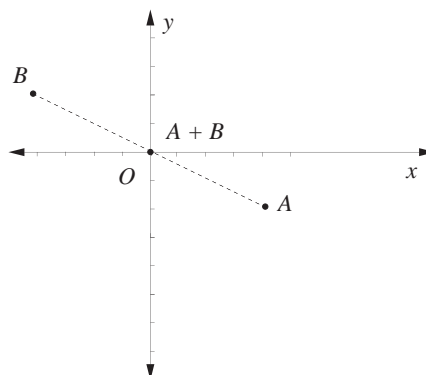
a. $A + B = (8, 7)$



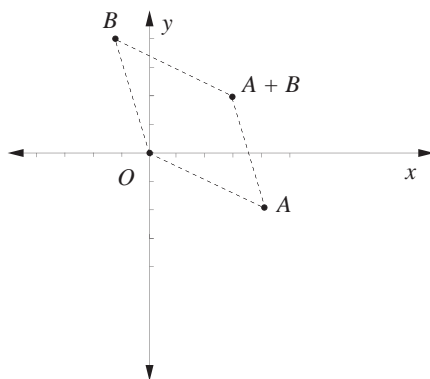
b. $A + B = (4, 4)$



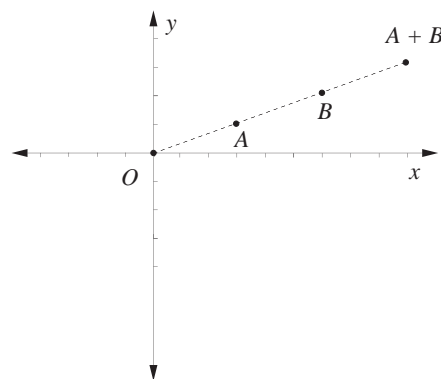
c. $A + B = (1, -7)$



d. $A + B = (0, 0)$



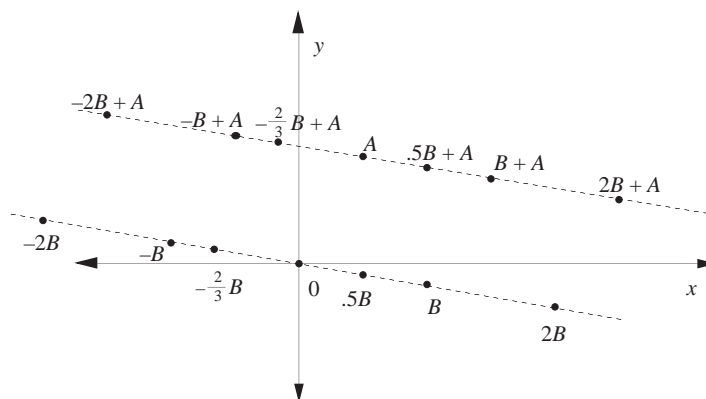
e. $A + B = (3, 2)$



f. $A + B = (9, 3)$

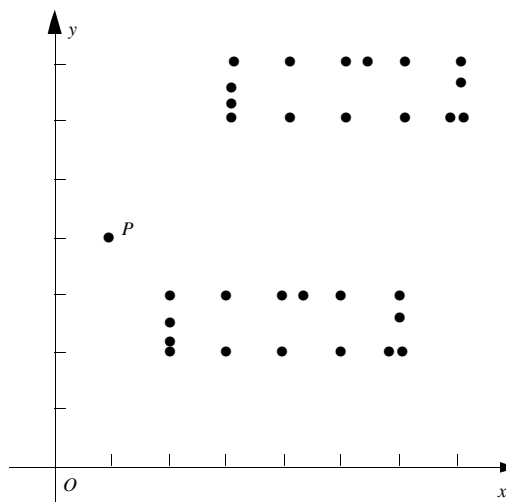
g. The location of $A + B$ is the fourth vertex of a parallelogram formed with the locations of A , B , and O . In some of the pictures the parallelogram formed by the four points is very clear; in a couple of them we have a special case, such as in f when O , A , B , and $A + B$ are all collinear. There the parallelogram has collapsed into a line segment, as it has in d where $A + B = O$.

Problem 2 (Student page 86) In the picture below, B , $B + A$, and the sum of several multiples of B with A have been plotted. If we plotted the sum of A and every possible multiple of B , we'd have two parallel lines; one through B and all of its multiples, and one through all the sums of A and all possible multiples of B .

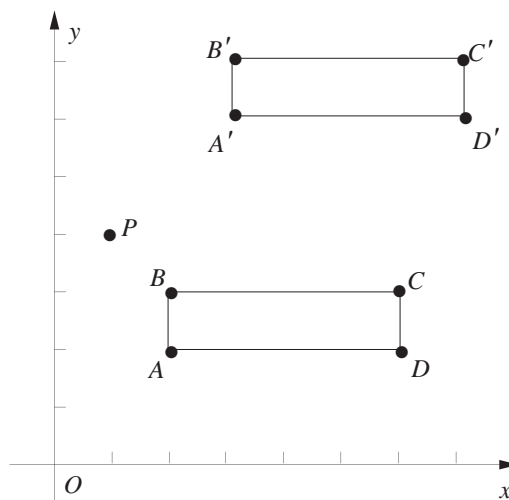


Unlike scaling, translating does not change the distance between corresponding points within the resulting figure.

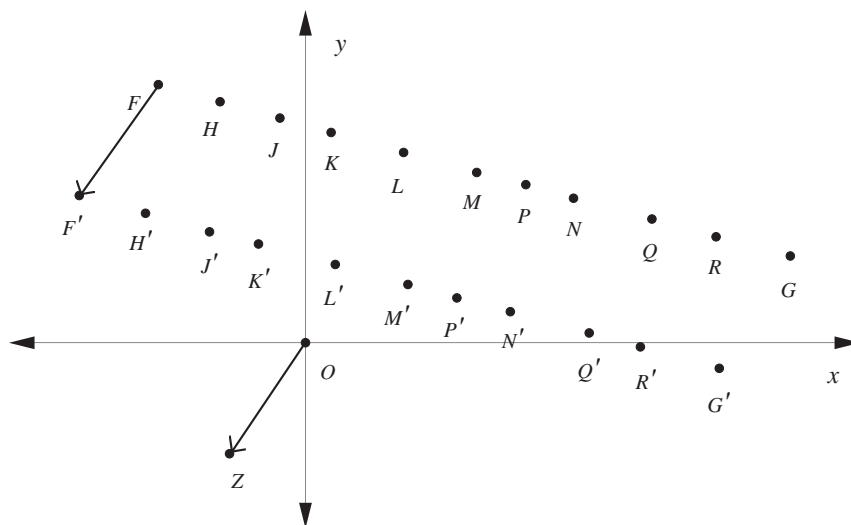
Problem 3 (Student page 86) The sum of P and each point in the rectangular set of points that's closest to the origin results in the second set of points—each point has been translated 1 unit to the right and 4 units up.



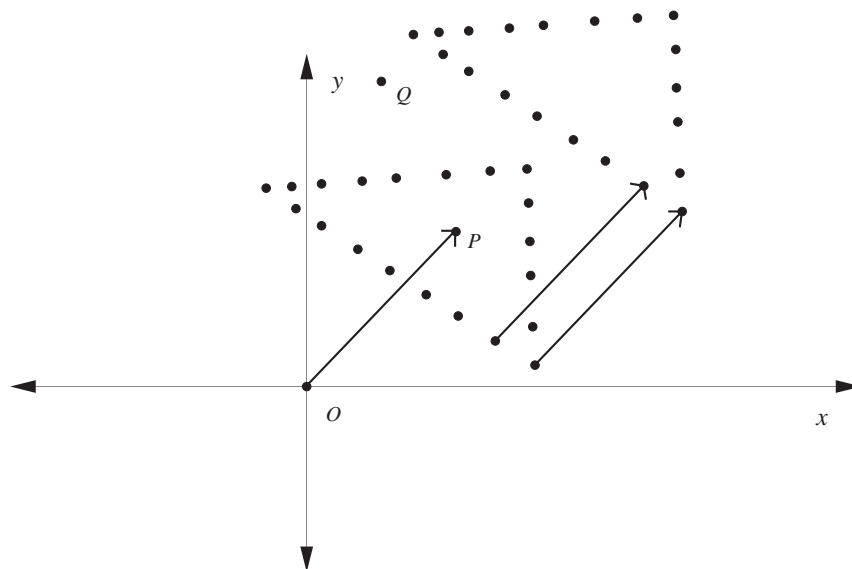
Problem 4 (Student page 87) $A' = A + P = (3, 6)$.



Problem 5 (Student page 87) No coordinates are given in the problem. You could either create your own scale, or you could draw a vector from O to Z ; then, using a straightedge or a ruler, mark off that same length and direction in order to add Z to points F through G , resulting in F' through G' . This is, of course, an approximate solution.



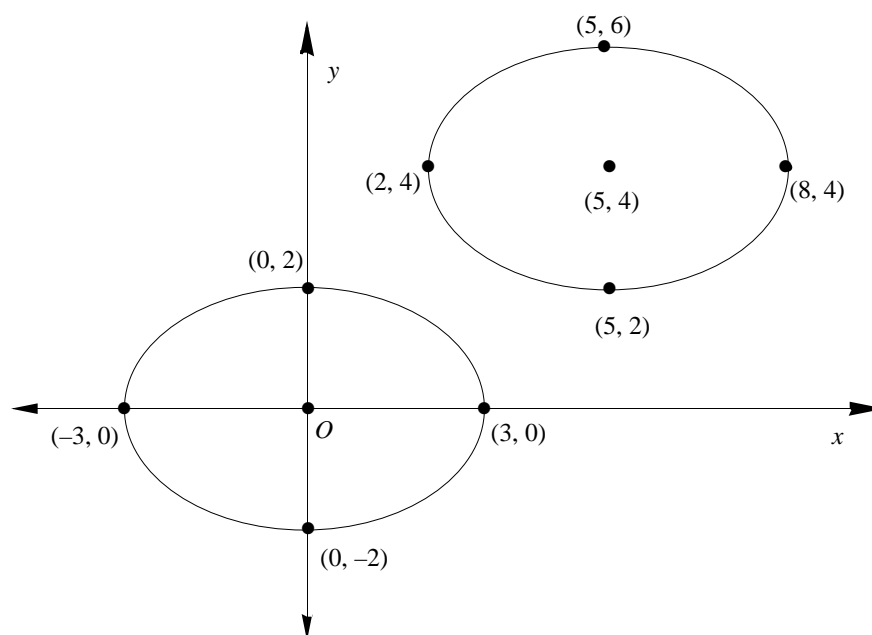
Problem 6 (Student page 87) Here the same kind of approximate solution has been done for point P .



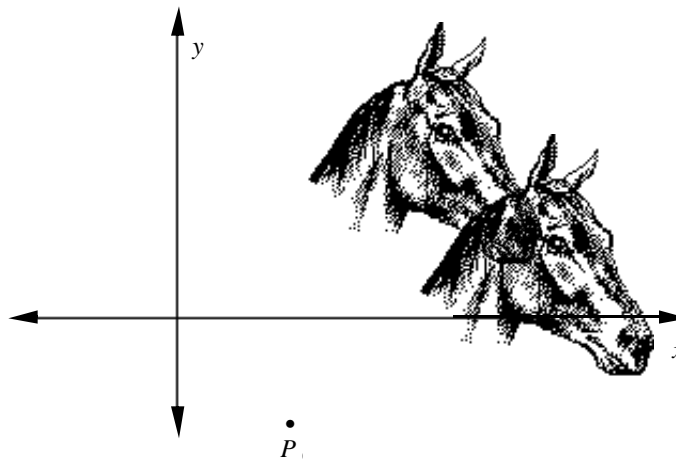
Problem 7 (Student page 88) Adding a point to all the points on a circle simply changes where the circle is centered. So adding $(8, -1)$ to the circle centered at the origin results in a circle of radius 6 centered at $(8, -1)$. Similarly for $(-4, 2)$ and $(0, 6)$ —the circle remains the same size (and shape), but changes location. Adding $(0, 0)$ to any point leaves it where it is, so you still have the same circle of radius 6 centered at the origin.

Like scaling by a factor of 1, adding the point $(0, 0)$ to any point or set of points is an identity transformation.

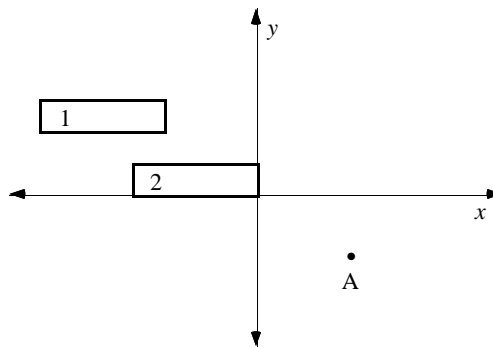
Problem 8 (Student page 88) The point $(3, 0)$ will be sent to $(8, 4)$, and $(0, -2)$ will be sent to $(5, 2)$.



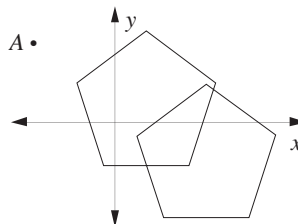
Problem 9 (Student page 88) Adding P results in a sketch that looks the same but has been moved down and to the right from the original.



Problem 10 (Student page 89) Adding point A to every point on rectangle 1 will result in rectangle 2:



Problem 11 (Student page 89) If you add A to every point on the pentagon that is mainly situated in the fourth quadrant, you'll get the pentagon that sits in all four quadrants.



Problem 12 (Student page 89) As mentioned in Problem 7, adding O (the point $(0, 0)$) to any point is the identity transformation; you get the very same pentagon.

Problem 13 (Student page 89) There are several ways to create the point $A + B$ using geometry software:

- Using an origin, A , and B , create the parallelogram with these points as vertices. The fourth vertex will be $A + B$.
- Create a vector from the origin to A , and translate B by that vector. The new point is $A + B$.
- Create a line segment from the origin to A . Create a line parallel to this through B . Mark off a length of OA along this line starting from B and taking into account whether you should go in the negative or positive direction.

Specific details for how to create each of these things can be found in the manual for your geometry software package.

- a. Once you have created $A + B$, you can animate A along various shapes. You will notice that $A + B$ traces out shapes congruent to the ones A travels along. If A animates on a segment, $A + B$ traces out a congruent and parallel segment. This is because, as A moves along the figure, $A + B$ is translating each of those points by B .
- b. It doesn't matter which of the points in our sketch we label A or B , so it doesn't matter which point we animate. $A + B$ will always trace out a shape congruent to (and translated from) the one B (or A) follows.

Problem 14 (Student page 90) Answers will vary. Here are some possibilities:

1. When you add two points, you add the corresponding coordinates.
2. When you add a point to all the points on a figure, you get a congruent figure somewhere else on the plane.
3. When you add a point to all the points on a figure, the orientation of the figure doesn't change.
4. The points O , A , B , and $A + B$ form a parallelogram.
5. To find $A + B$ if you don't know coordinates, you can copy the vector from O to A onto B . The new head is at $A + B$.

Problem 15 (Student page 90) Looking at the picture, it appears that A can be translated onto D (and likewise D onto A) since they seem to be congruent triangles with the same orientation (corresponding sides are parallel). Neither can be translated onto B or C because they are much larger, and neither can be translated onto E because it has been flipped.

Problem 16 (Student page 90) If it's possible to add a point to every point on one figure to get another figure, then the two figures must be congruent. Also, they must have the same orientation—corresponding segments (or figures with straight segments) must be parallel.

Problem 17 (Student page 90) The point $B + 3A$ is at $(6, 10, 10)$; it's 10 feet off the ground. If anyone can reach it (if your ceilings are high enough), he or she will have to stand on something!

The answer to this problem depends upon looking at the drawing and making assumptions based on how it looks rather than proving it mathematically. We can eliminate some cases and say that others look promising, but without more than this picture, we can't prove a correct answer.

Each of the points is of the form $3 + cA$ for some number c . We know that all the points cA lie on a line through the origin, so we have just translated that whole line by B . It is a line through the points B (remember $B + OA$) and $A + B$.

Problem 18 (Student page 91)

- a. $(5, 8)$
- b. $(3, 16)$
- c. $(5, -6)$
- d. $(\frac{7}{2}, -7)$
- e. $(-\frac{5}{2}, 17)$
- f. $(-k + 3j, 4k + 2j)$

Problem 19 (Student page 91)

- a. You want to find numbers a and b such that $a(-1, 4) + b(3, 2) = (-6, 10)$. You could plot A and B , locate $(-6, 10)$, and draw lines through the origin and A and the origin and B . Then you would find which multiples of A and B will make the second and third vertices of a parallelogram with the origin and $(-6, 10)$.

Alternatively, you could look at the x - and y -coordinates algebraically, using a system of equations:

$$-a + 3b = -6, \text{ so } a = 6 + 3b, \text{ and } 4a + 2b = 10.$$

By substitution, $4(6 + 3b) + 2b = 10$, and $24 + 14b = 10$.

So $b = -1$, and $a = 3$; thus $3A - B = (-6, 10)$.

- b. Yes, every point can be written as $aA + bB$ for some choice of a and b . (In this case we say that A and B span the plane.) Here is an algebraic proof:

Let (x, y) be any point in the xy -plane. We want to find a and b such that $aA + bB = (x, y)$. As in part a, we write a system of equations. From the x -coordinates, we have

$$(1) \quad -a + 3b = x,$$

and from the y -coordinates we have

$$(2) \quad 4a + 2b = y.$$

Multiply equation (1) by 4 and add the result to equation (2) to obtain

$$\begin{aligned} 14b &= 4x + y \\ b &= \frac{4x + y}{14}. \end{aligned}$$

Now substitute this expression for b into equation (1) and solve for a :

$$\begin{aligned} -a + 3\left(\frac{4x + y}{14}\right) &= x \\ a &= 3\left(\frac{4x + y}{14}\right) - x \\ &= \frac{12x + 3y}{14} - \frac{14x}{14} \\ &= \frac{3y - 2x}{14}. \end{aligned}$$

Thus, for any x and y , we can find an a and b that will work. There are no values for x and y that make any of the steps invalid, so we can write $P = aA + bB$ for every point in the plane.

Problem 20 (Student page 91) $A = (6, 4)$ and $B = (3, 2)$. We want $aA + bB = (9, 6)$ for some a and b . Insert the x -coordinates of A and B to get

$$\begin{aligned} 6a + 3b &= 9 \\ b &= 3 - 2a, \end{aligned}$$

and for the y -coordinates,

$$\begin{aligned} 4a + 2b &= 6 \\ b &= 3 - 2a. \end{aligned}$$

Many people will notice the solution of $a = b = 1$ quickly, but what about $a = 0, b = 3$?

Because these give the same equation, we can pick *any* value for a , and we'll get a corresponding value for b .

But, if instead we want $aA + bB = (-9, 4)$, it's a different story. For the x -coordinates:

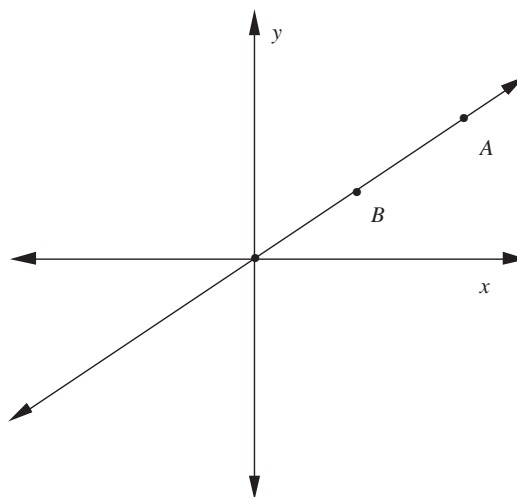
$$6a + 3b = -9 \Rightarrow b = -3 - 2a,$$

and for the y -coordinates

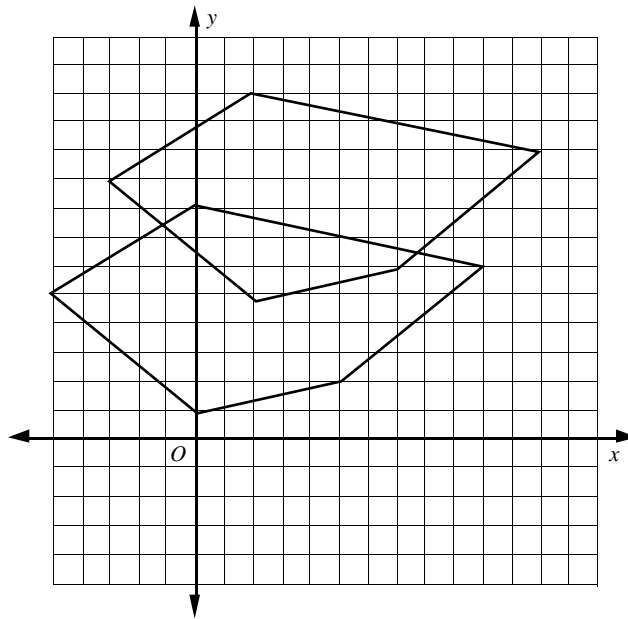
$$4a + 2b = 4 \Rightarrow b = 2 - 2a.$$

But $-3 - 2a \neq 2 - 2a$ since $-3 \neq 2$. There is no simultaneous solution for both the x -coordinate and the y -coordinate. Thus, there are no numbers a and b such that $aA + bB = (-9, 4)$.

Because A and B are multiples of each other ($A = 2B$), they are on the same line through the origin. Every multiple of A is on that line, and so is every multiple of B . Thus, every sum $aA + bB$ is really the same as a multiple of B , ($aA = 2aB$, so $aA + bB = (2a + b)B$); and each of these sums will represent points on that same line through the origin.



Problem 21 (Student page 95) All the points on the original pentagon have been translated by $(-2, -4)$.



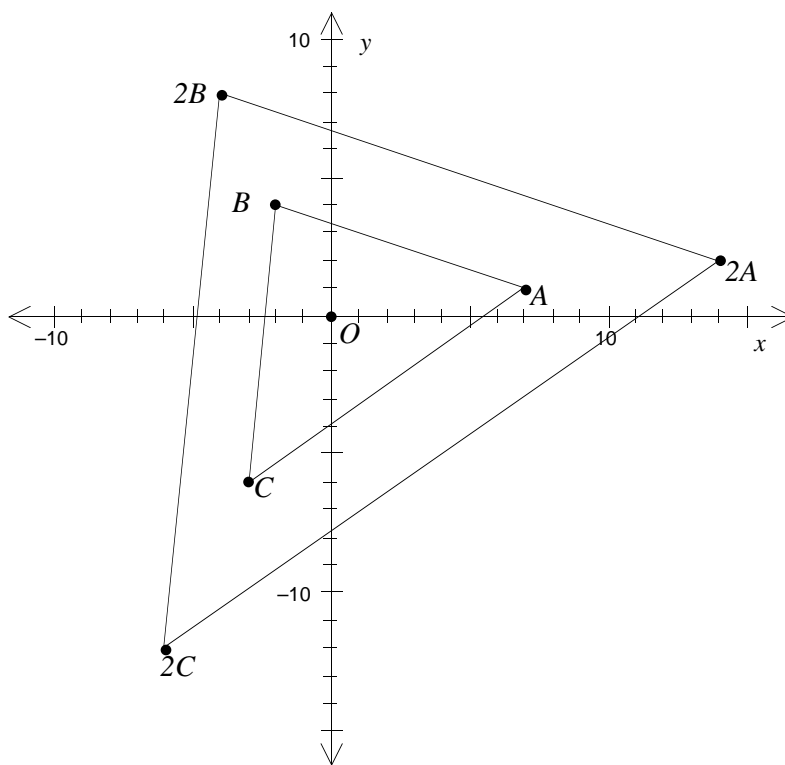
Problem 22 (Student page 95)

- a. $(-2, 8)$
- b. $(2, -8)$
- c. Possible methods:
 - You counted how many over and up (or down) between corresponding vertices; or
 - You drew an arrow (vector) between 2 corresponding vertices, moved that arrow to the origin, and checked where the tip was.

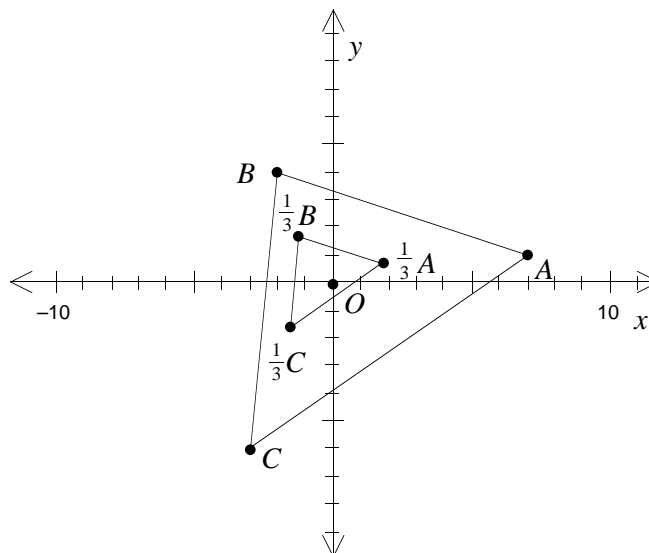
MAKING THINGS PRECISE

Problem 1 (*Student page 98*) The distance from O to cA is *always* $|c|$ times the distance from O to A . So if $c = 0$, then the distance from O to cA is zero, which means that cA must be at O . Likewise, if c is either 1 or -1 , then the distance from O to cA is the same as the distance from O to A . There are many points that fit this description (how many?), but only two of them lie along the line through O and A . When $c = 1$, $cA = A$.

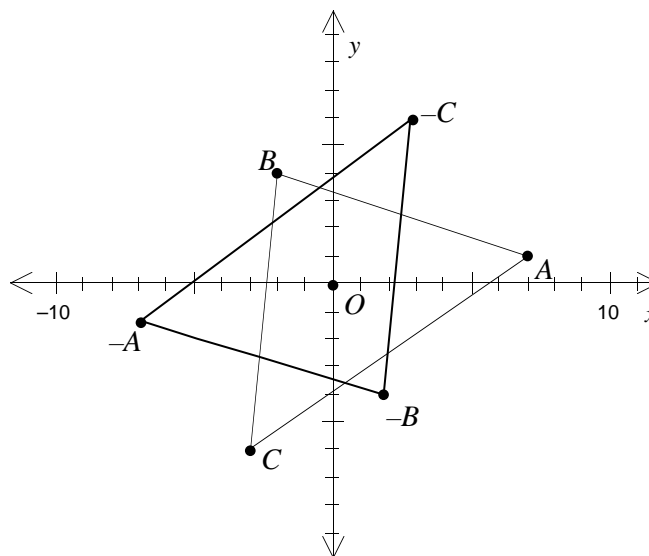
Problem 2 (*Student page 98*)



a. $\triangle(2A)(2B)(2C)$

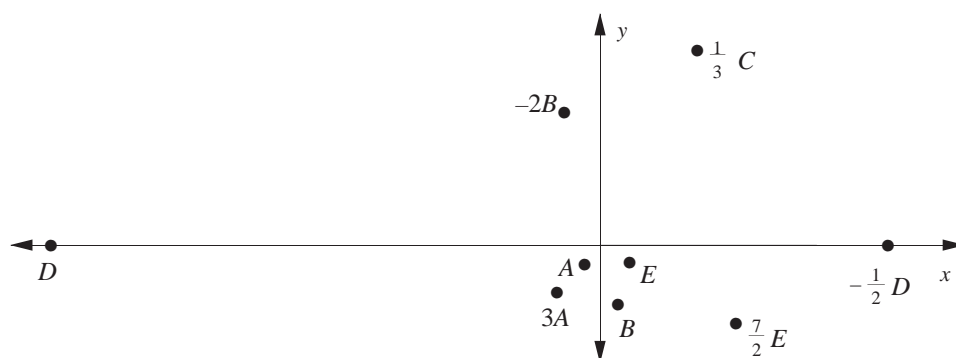


b. $\triangle(\frac{1}{3}A)(\frac{1}{3}B)(\frac{1}{3}C)$



c. $\triangle(-A)(-B)(-C)$

Problem 3 (Student page 98)

For Discussion (Student page 100)

1. cB is a point.
2. AB is a number.
3. $c(AB)$ is a number.
4. $k(OA)$ is a number.
5. $A(BC)$ is meaningless or a point.
6. x is a number.
7. xX is a point.
8. XX is a number (in fact 0).
9. Bc is meaningless or a point.

The only questionable ones are items 5 and 9. If we decide that cB is the same as Bc , then these are points. That's unusual notation, though, so you might decide they are meaningless.

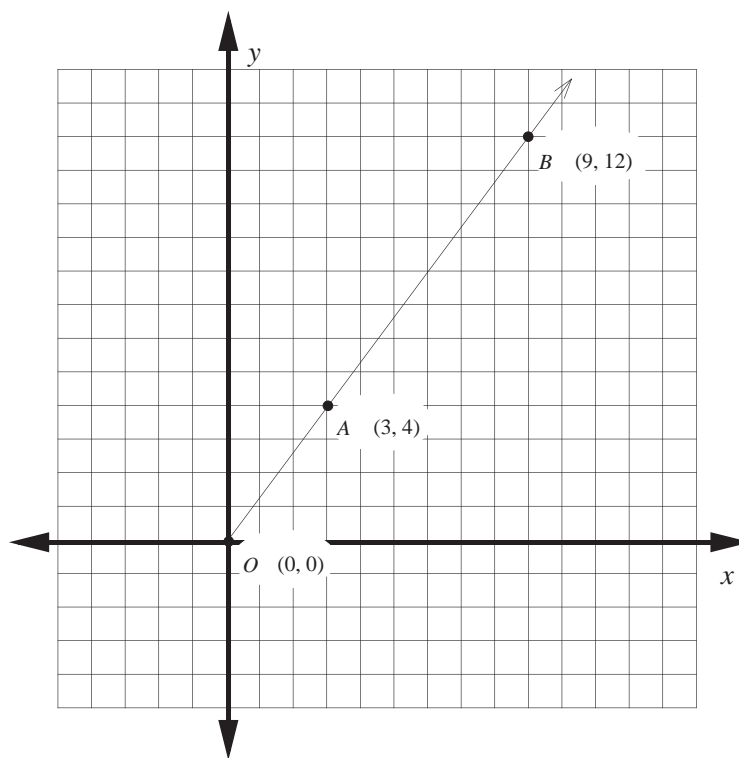
Problem 4 (Student page 101) You can work right with the numbers, calculating like this:

$$OA = \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{25} = 5$$

$$AB = \sqrt{(9-3)^2 + (12-4)^2} = \sqrt{100} = 10 \quad \text{and}$$

$$OB = \sqrt{(9-0)^2 + (12-0)^2} = \sqrt{225} = 15.$$

So, $OA + AB = OB$ and $OB = 3OA$.



But, since this is supposed to be preparation for the proof, think about saving the simplifications until the end and calculating like this:

$$\begin{aligned}
 OA &= \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{25} = 5 \\
 AB &= \sqrt{(3 \cdot 3 - 3)^2 + (3 \cdot 4 - 4)^2} \\
 &= \sqrt{[(3-1) \cdot 3]^2 + [(3-1) \cdot 4]^2} \\
 &= \sqrt{(3-1)^2 \cdot 3^2 + (3-1)^2 \cdot 4^2} \\
 &= \sqrt{(3-1)^2 \cdot (3^2 + 4^2)} \\
 &= \sqrt{(3-1)^2} \sqrt{3^2 + 4^2} \\
 &= (3-1) \sqrt{25} \\
 &= 2 \cdot 5
 \end{aligned}$$

and

$$\begin{aligned}
 OB &= \sqrt{(3 \cdot 3)^2 + (3 \cdot 4)^2} \\
 &= \sqrt{3^2 \cdot 3^2 + 3^2 \cdot 4^2} \\
 &= \sqrt{3^2(3^2 + 4^2)} \\
 &= \sqrt{3^2} \cdot \sqrt{3^2 + 4^2} \\
 &= 3 \cdot \sqrt{25} \\
 &= 3 \cdot 5.
 \end{aligned}$$

You are looking for the **form** of the calculation, so you can turn the numerical calculation into an algebraic one.

So,

$$\begin{aligned}
 OA + AB &= 5 + 2 \cdot 5 \\
 &= 3 \cdot 5 \\
 &= OB,
 \end{aligned}$$

and $OB = 3OA$.

Problem 5 (Student page 101)

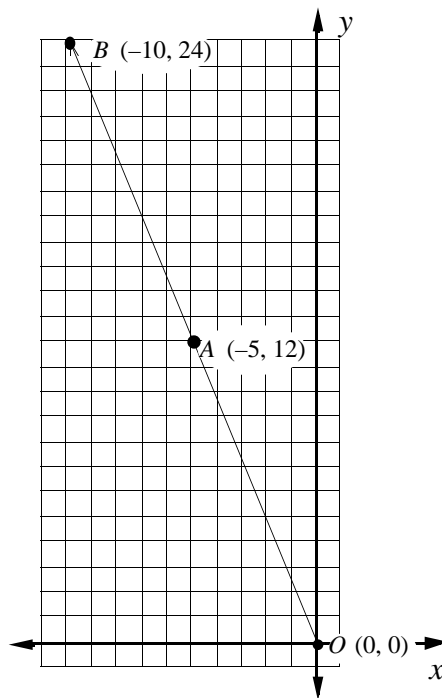
You can “do the numbers”:

$$OA = \sqrt{(-5 - 0)^2 + (12 - 0)^2} = \sqrt{169} = 13$$

$$AB = \sqrt{[-10 - (-5)]^2 + (24 - 12)^2} = \sqrt{169} = 13$$

$$OB = \sqrt{(-10 - 0)^2 + (24 - 0)^2} = \sqrt{676} = 26.$$

So, $OA + AB = OB$, and $OB = 2OA$.



Or you gear up for a generic calculation:

$$OA = \sqrt{(-5 - 0)^2 + (12 - 0)^2} = \sqrt{169} = 13$$

$$\begin{aligned} AB &= \sqrt{[2 \cdot (-5) - (-5)]^2 + (2 \cdot 12 - 12)^2} \\ &= \sqrt{[(2 - 1) \cdot (-5)]^2 + [(2 - 1) \cdot 12]^2} \\ &= \sqrt{(2 - 1)^2 \cdot (-5)^2 + (2 - 1)^2 \cdot 12^2} \\ &= \sqrt{(2 - 1)^2 \cdot [(-5)^2 + 12^2]} \\ &= \sqrt{(2 - 1)^2} \sqrt{(-5)^2 + 12^2} \\ &= (2 - 1) \sqrt{169} \\ &= 1 \cdot 13 \end{aligned}$$

and

$$\begin{aligned} OB &= \sqrt{[2 \cdot (-5)]^2 + (2 \cdot 12)^2} \\ &= \sqrt{2^2 \cdot (-5)^2 + 2^2 \cdot 12^2} \\ &= \sqrt{2^2 [(-5)^2 + 12^2]} \\ &= \sqrt{2^2} \cdot \sqrt{(-5)^2 + 12^2} \\ &= 2 \cdot \sqrt{169} \\ &= 2 \cdot 13. \end{aligned}$$

So,

$$\begin{aligned} OA + AB &= 13 + 1 \cdot 13 \\ &= 2 \cdot 13 \\ &= OB, \end{aligned}$$

and $OB = 2OA$.

Problem 6 (Student page 101)

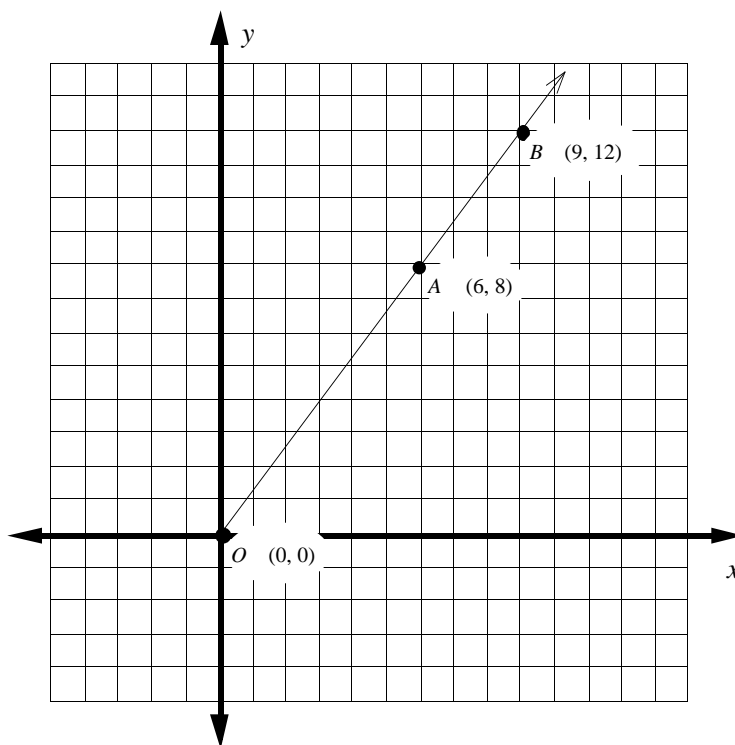
Work with the numbers:

$$OA = \sqrt{(6-0)^2 + (8-0)^2} = \sqrt{100} = 10$$

$$AB = \sqrt{(9-6)^2 + (12-8)^2} = \sqrt{25} = 5$$

$$OB = \sqrt{(9-0)^2 + (12-0)^2} = \sqrt{225} = 15.$$

So, $OA + AB = OB$ and $OB = 1.5OA$.



Or get ready for the algebra:

$$OA = \sqrt{(6-0)^2 + (8-0)^2} = \sqrt{100} = 10$$

$$\begin{aligned} AB &= \sqrt{[1.5 \cdot 6 - 6]^2 + [1.5 \cdot 8 - 8]^2} \\ &= \sqrt{[(1.5 - 1) \cdot 6]^2 + [(1.5 - 1) \cdot 8]^2} \\ &= \sqrt{(1.5 - 1)^2 \cdot 6^2 + (1.5 - 1)^2 \cdot 8^2} \\ &= \sqrt{(1.5 - 1)^2 \cdot (6^2 + 8^2)} \\ &= \sqrt{(1.5 - 1)^2} \sqrt{6^2 + 8^2} \\ &= (1.5 - 1) \sqrt{100} \\ &= 0.5 \cdot 10 \end{aligned}$$

and

$$\begin{aligned} OB &= \sqrt{(1.5 \cdot 6)^2 + (1.5 \cdot 8)^2} \\ &= \sqrt{1.5^2 \cdot 6^2 + 1.5^2 \cdot 8^2} \\ &= \sqrt{1.5^2(6^2 + 8^2)} \\ &= \sqrt{1.5^2} \cdot \sqrt{6^2 + 8^2} \\ &= 1.5 \cdot \sqrt{100} \\ &= 1.5 \cdot 10. \end{aligned}$$

So,

$$\begin{aligned} OA + AB &= 10 + 0.5 \cdot 10 \\ &= 1.5 \cdot 10 \\ &= OB, \end{aligned}$$

and $OB = 1.5OA$.

Problem 7 (Student page 102) Let's go right to the second type of calculation:

$$\begin{aligned}
 OA &= \sqrt{(3-0)^2 + (8-0)^2} = \sqrt{73} \\
 AB &= \sqrt{(4 \cdot 3 - 3)^2 + (4 \cdot 8 - 8)^2} \\
 &= \sqrt{[(4-1) \cdot 3]^2 + [(4-1) \cdot 8]^2} \\
 &= \sqrt{(4-1)^2 \cdot 3^2 + (4-1)^2 \cdot 8^2} \\
 &= \sqrt{(4-1)^2 \cdot (3^2 + 8^2)} \\
 &= \sqrt{(4-1)^2} \sqrt{3^2 + 8^2} \\
 &= (4-1)\sqrt{73} \\
 &= 3\sqrt{73}
 \end{aligned}$$

and

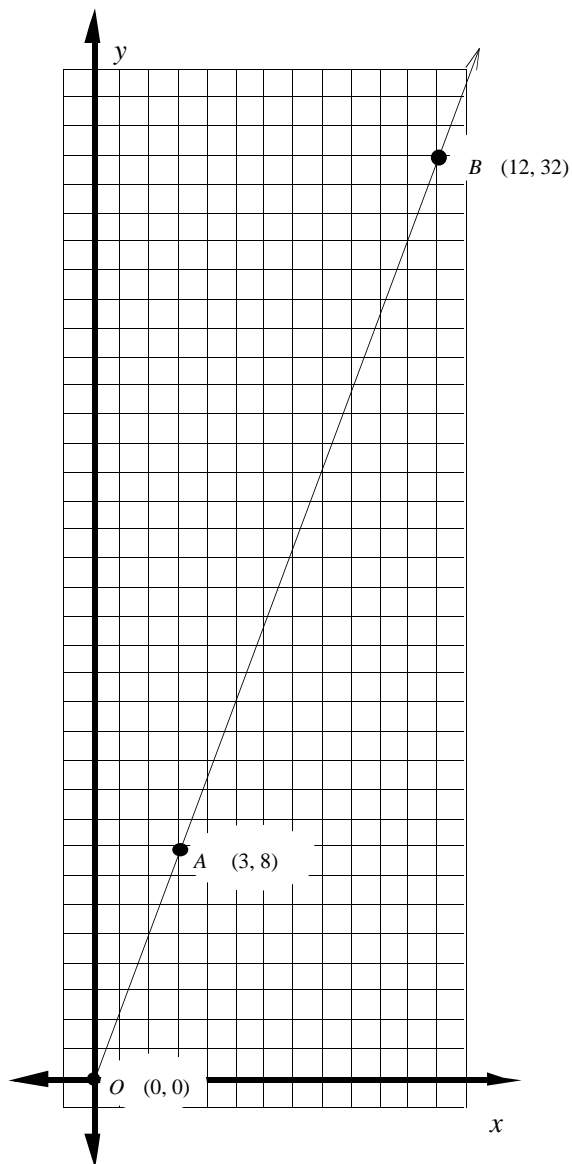
$$\begin{aligned}
 OB &= \sqrt{(4 \cdot 3)^2 + (4 \cdot 8)^2} \\
 &= \sqrt{4^2 \cdot 3^2 + 4^2 \cdot 8^2} \\
 &= \sqrt{4^2(3^2 + 8^2)} \\
 &= \sqrt{4^2} \cdot \sqrt{3^2 + 8^2} \\
 &= 4 \cdot \sqrt{73}.
 \end{aligned}$$

In many ways, using irrational numbers like $\sqrt{73}$ makes it easier to see the essence of the calculation, because you can't simplify the numbers.

So,

$$\begin{aligned}
 OA + AB &= \sqrt{73} + 3\sqrt{73} \\
 &= 4\sqrt{73} \\
 &= OB,
 \end{aligned}$$

and $OB = 4OA$.



This is not *exactly* the same as the proof in the Student Module. Here, $A = (x, y)$; in the Student Module, $A = (a_1, a_2)$. Which notation makes it clearer to you?

Treat $\sqrt{x^2 + y^2}$ the same way as you treated $\sqrt{73}$ in Problem 7.

Problem 8 (Student page 102) By now, you can almost predict how it will go. What's here is just the proof of Theorem 5.3:

$$\begin{aligned} OA &= \sqrt{(x-0)^2 + (y-0)^2} \\ &= \sqrt{x^2 + y^2} \\ AB &= \sqrt{(cx-x)^2 + (cy-y)^2} \\ &= \sqrt{[(c-1)x]^2 + [(c-1)y]^2} \\ &= \sqrt{(c-1)^2x^2 + (c-1)^2y^2} \\ &= \sqrt{(c-1)^2(x^2 + y^2)} \\ &= \sqrt{(c-1)^2} \sqrt{x^2 + y^2} \\ &= (c-1)\sqrt{x^2 + y^2} \end{aligned}$$

and

$$\begin{aligned} OB &= \sqrt{(cx)^2 + (cy)^2} \\ &= \sqrt{c^2x^2 + c^2y^2} \\ &= \sqrt{c^2(x^2 + y^2)} \\ &= \sqrt{c^2} \sqrt{x^2 + y^2} \\ &= c\sqrt{x^2 + y^2}. \end{aligned}$$

So,

$$\begin{aligned} OA + AB &= \sqrt{x^2 + y^2} + (c-1)\sqrt{x^2 + y^2} \\ &= [1 + (c-1)]\sqrt{x^2 + y^2} \\ &= c\sqrt{x^2 + y^2} \\ &= OB, \end{aligned}$$

and $OB = cOA$.

Problem 9 (Student page 102)

$$OA + AB = \sqrt{a_1^2 + a_2^2} + \sqrt{(ca_1 - a_1)^2 + (ca_2 - a_2)^2}$$

We used the distance formula.

$$= \sqrt{a_1^2 + a_2^2} + \sqrt{[a_1(c - 1)]^2 + [a_2(c - 1)]^2}$$

We used the distributive law to factor out a_1 and a_2 .

$$= \sqrt{a_1^2 + a_2^2} + \sqrt{a_1^2(c - 1)^2 + a_2^2(c - 1)^2}$$

The square of the products is the product of the squares.

$$= \sqrt{a_1^2 + a_2^2} + \sqrt{(c - 1)^2(a_1^2 + a_2^2)}$$

We used the distributive law to factor out $(c - 1)^2$.

$$= \sqrt{a_1^2 + a_2^2} + \sqrt{(c - 1)^2} \sqrt{(a_1^2 + a_2^2)}$$

The square root of a product is the product of the square roots
(provided everything is positive).

$$= \sqrt{a_1^2 + a_2^2} + (c - 1) \sqrt{(a_1^2 + a_2^2)}$$

We took the positive square root of a perfect square ($c \geq 1$).

$$= (1 + c - 1) \sqrt{a_1^2 + a_2^2}$$

We used the distributive law to factor out $\sqrt{a_1^2 + a_2^2}$.

$$= c \sqrt{a_1^2 + a_2^2}$$

We just used arithmetic.

$$OB = \sqrt{(ca_1)^2 + (ca_2)^2}$$

We used the distance formula.

$$= \sqrt{c^2 a_1^2 + c^2 a_2^2}$$

The square of the products is the product of the squares.

$$= \sqrt{c^2(a_1^2 + a_2^2)}$$

We used the distributive law to factor out c^2 .

$$= \sqrt{c^2} \sqrt{a_1^2 + a_2^2}$$

The square root of a product is the product of the square roots
(provided everything is positive).

$$= c \sqrt{a_1^2 + a_2^2}$$

We took the positive square root of a perfect square ($c \geq 1$).

Problem 10 (Student page 102) For certain values of c , there are two places where things go wrong: They are

From the calculation of $OA + AB$:

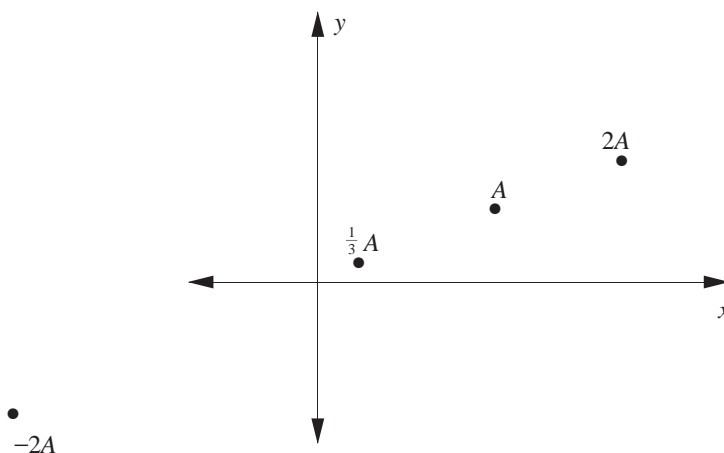
$$\sqrt{a_1^2 + a_2^2} + \sqrt{(c-1)^2} \sqrt{a_1^2 + a_2^2} = \sqrt{a_1^2 + a_2^2} + (c-1) \sqrt{a_1^2 + a_2^2}.$$

From the calculation of OB :

$$\sqrt{c^2} \sqrt{a_1^2 + a_2^2} = c \sqrt{a_1^2 + a_2^2}.$$

In the first case, $\sqrt{(c-1)^2}$ is replaced with $c-1$, and in the second case, $\sqrt{c^2}$ is replaced by c . When $c < 1$, $(c-1)$ gives the negative square root, which doesn't make sense for distance. Similarly, when $c < 0$, c gives the negative square root.

Problem 11 (Student page 103) In this case, B is a small (less than 1) positive multiple of A . It's like the $\frac{1}{3}A$ in this picture:



Notice the crucial steps.

Why is $\sqrt{(c-1)^2} = 1-c$?

Why is $\sqrt{c^2} = c$?

So, show that $OB + BA = OA$ and $OB = cOA$. Here we go again (let $A = (x, y)$):

$$\begin{aligned} OA &= \sqrt{x^2 + y^2} \\ BA &= \sqrt{(cx - x)^2 + (cy - y)^2} \\ &= \sqrt{[(c-1)x]^2 + [(c-1)y]^2} \\ &= \sqrt{(c-1)^2x^2 + (c-1)^2y^2} \\ &= \sqrt{(c-1)^2(x^2 + y^2)} \\ &= \sqrt{(c-1)^2} \sqrt{x^2 + y^2} \\ &= (1-c) \sqrt{x^2 + y^2} \end{aligned}$$

and

$$\begin{aligned} OB &= \sqrt{(cx)^2 + (cy)^2} \\ &= \sqrt{c^2x^2 + c^2y^2} \\ &= \sqrt{c^2(x^2 + y^2)} \\ &= \sqrt{c^2} \sqrt{x^2 + y^2} \\ &= c \sqrt{x^2 + y^2}. \end{aligned}$$

So,

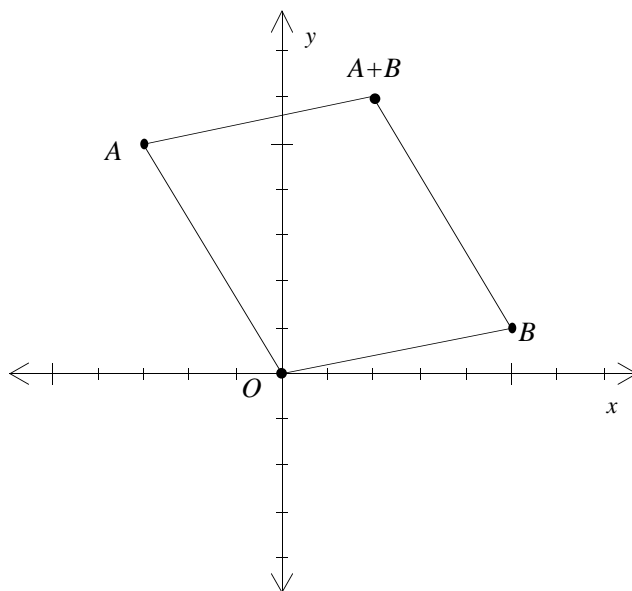
$$\begin{aligned} OB + BA &= c \sqrt{x^2 + y^2} + (1-c) \sqrt{x^2 + y^2} \\ &= [c + (1-c)] \sqrt{x^2 + y^2} \\ &= \sqrt{x^2 + y^2} \\ &= OA, \end{aligned}$$

and $OB = cOA$.

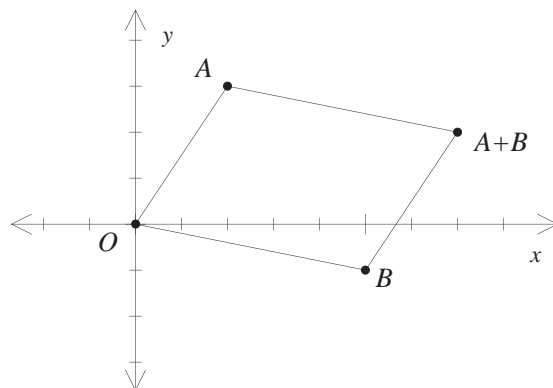
Problem 12 (Student page 103) You could go through the same process as in Problem 11, and that would work. You would show this time that $BO + OA = BA$. Or, you could use the habit of mind that recycles old work. Argue like this:

If $c < 0$, then $-c > 0$. So, we know the story with $-cA$: If $B' = -cA$, B' is collinear with A and the origin and $OB' = -cOA$. But B' is related to B in an extremely simple way: B' is just the “mirror image” of B through the origin. So, information about B' translates into information about B . More precisely, since $\overline{OB'} \cong \overline{OB}$, $OB' = -cOA$. Since B , O , and B' are collinear and B' , O , and A are collinear, B , O , and A must be collinear.

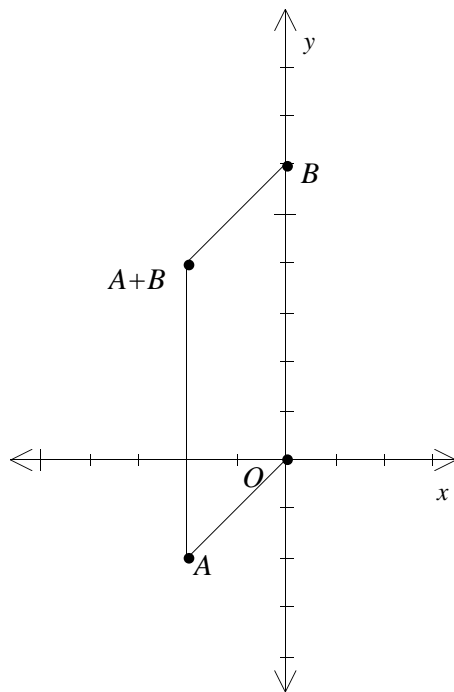
Problems 13–14 (Student pages 103–104)



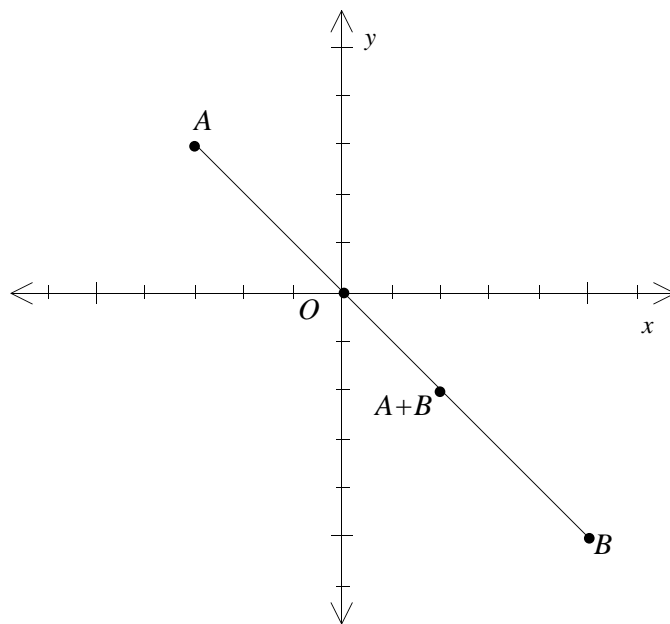
a. $A + B = (2, 6)$



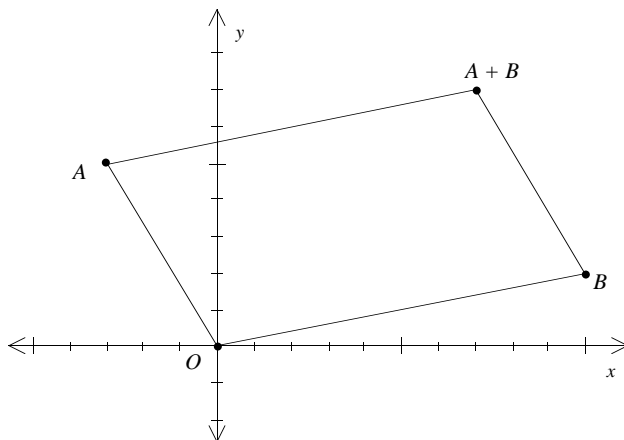
b. $A + B = (7, 2)$



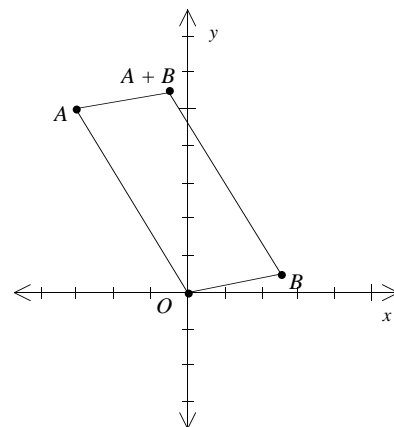
c. $A + B = (-2, 4)$



d. $A + B = (2, -2)$



e. $A + B = (7, 7)$



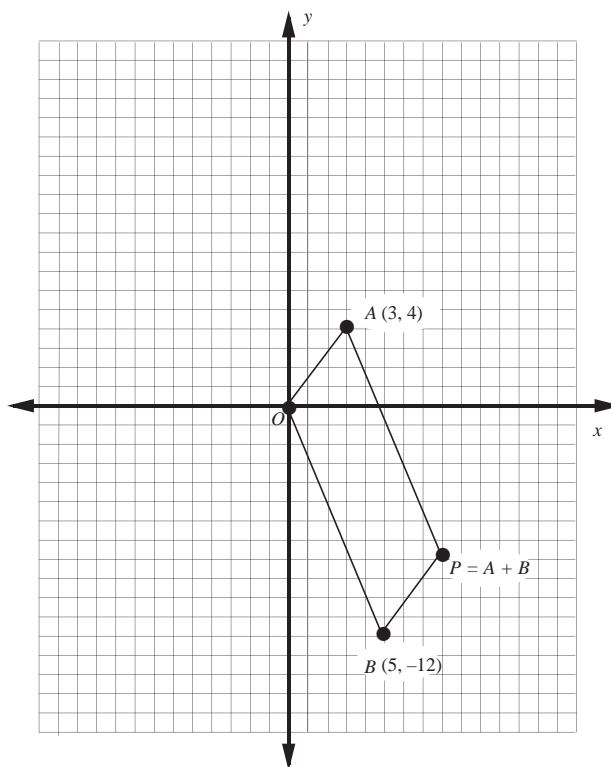
f. $A + B = (-\frac{1}{2}, \frac{11}{2})$

When you connect O to A and B and $A + B$ to A and B you get a parallelogram (unless A , B , and O are collinear, in which case all four points are collinear and you get a segment).

Problem 15 (Student page 104) Here is one way to make the theorem clearer:

$A + B$ is the fourth vertex of the parallelogram whose other three vertices are A , O , and B , and which has $\angle AOB$ as one of its angles.

Problem 16 (Student page 105) Problems 16–19 are designed to help you gear up algebraically for the proof of Theorem 5.4. Once again, the idea is to concentrate on the form of the calculation, not to make use of the particular numbers in the problem.



$$OA = \sqrt{3^2 + 4^2} = 5$$

$$\begin{aligned} BP &= \sqrt{[(3 + 5) - 5]^2 + [(4 + (-12)) - (-12)]^2} \\ &= \sqrt{3^2 + 4^2} = 5 \end{aligned}$$

So, $OA = BP$.

The other pair of sides is also the same length:

$$OB = \sqrt{5^2 + (-12)^2} = 13$$

$$\begin{aligned} AP &= \sqrt{[(5 + 3) - 3]^2 + [(-12 + 4) - 4]^2} \\ &= \sqrt{5^2 + (-12)^2} = 13, \end{aligned}$$

so $OB = AP$.

Problem 17 (Student page 105)

$$OA = \sqrt{8^2 + 15^2} = 17$$

$$\begin{aligned} BP &= \sqrt{[(8 + (-4) - (-4))]^2 + [(15 + 3) - 3]^2} \\ &= \sqrt{8^2 + 15^2} = 17 \end{aligned}$$

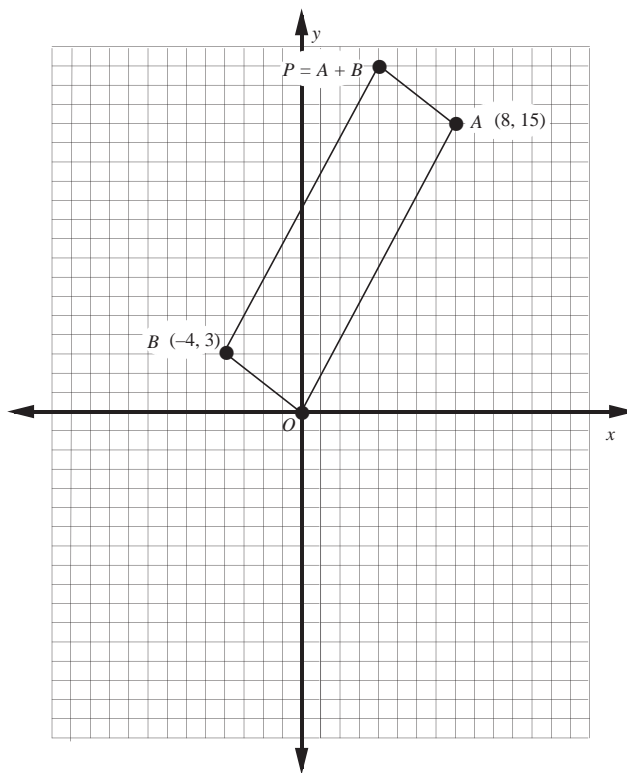
So, $OA = BP$.

The other pair of sides is also the same length:

$$OB = \sqrt{(-4)^2 + (3)^2} = 5$$

$$\begin{aligned} AP &= \sqrt{[(8 + (-4)) - 8]^2 + [(15 + 3) - 15]^2} \\ &= \sqrt{(-4)^2 + (3)^2} = 5. \end{aligned}$$

So, $OB = AP$.

**Problem 18** (Student page 105)

$$OB = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$\begin{aligned} AP &= \sqrt{[(3+8)-8]^2 + [(1+6)-6]^2} \\ &= \sqrt{3^2 + 1^2} = \sqrt{10} \end{aligned}$$

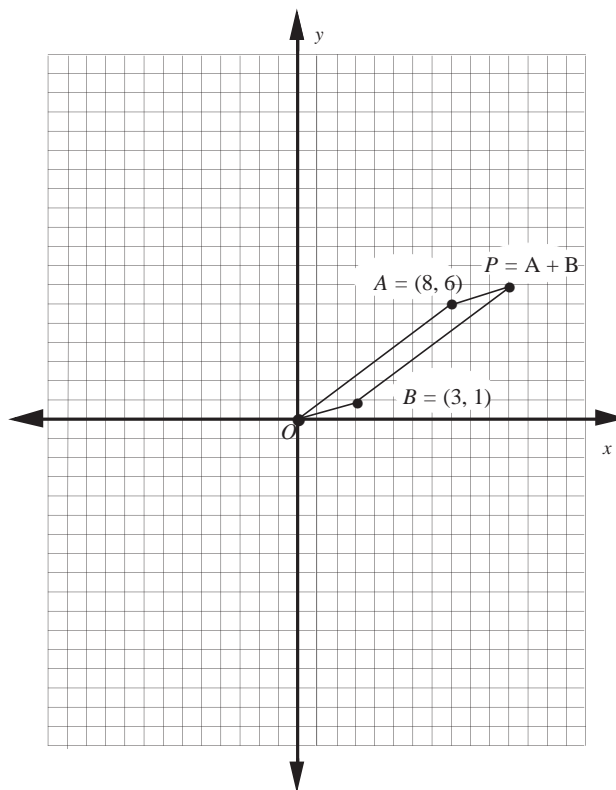
So, $OB = AP$.

The other pair of sides is also the same length:

$$OA = \sqrt{8^2 + 6^2} = 10$$

$$\begin{aligned} BP &= \sqrt{[(3+8)-3]^2 + [(1+6)-1]^2} \\ &= \sqrt{8^2 + 6^2} = 10, \end{aligned}$$

so $OA = BP$.



Problem 19 (Student page 105) This is essentially the same as the proof of Theorem 5.4 on pages 105–106 of the Student Module.

Problem 20 (Student page 106)

$$OB = \sqrt{b_1^2 + b_2^2}$$

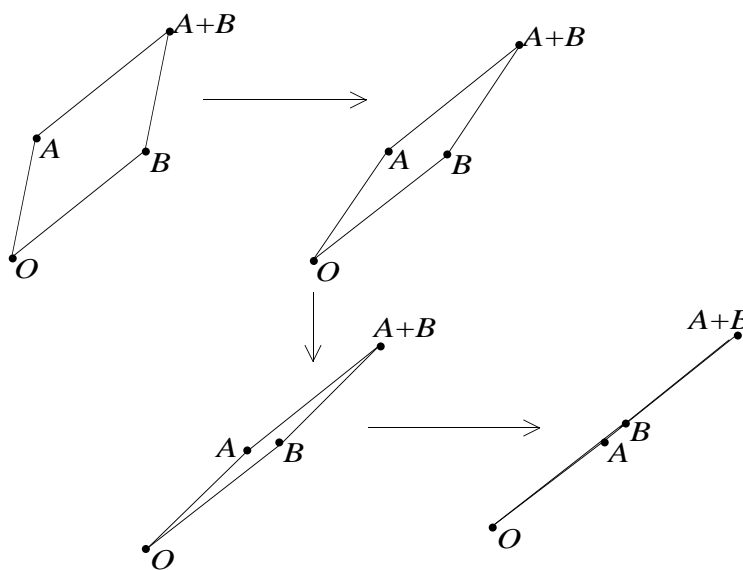
$$\begin{aligned} PA &= \sqrt{[(b_1 + a_1) - a_1]^2 + [(b_2 + a_2) - a_2]^2} \\ &= \sqrt{b_1^2 + b_2^2} \end{aligned}$$

So, $OB = PA$.

Problem 21 (Student page 106) “Does Theorem 5.4 hold if O , A , and B are collinear?”

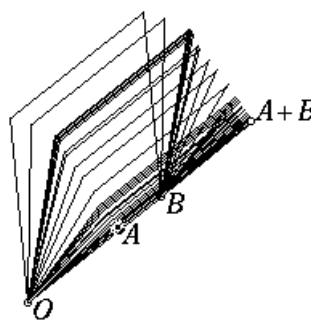
Well, sort of.

If O , A , and B are collinear, $A + B$ is on \overleftrightarrow{AB} . Think of a parallelogram “collapsing” by making $\angle AOB$ smaller and smaller.



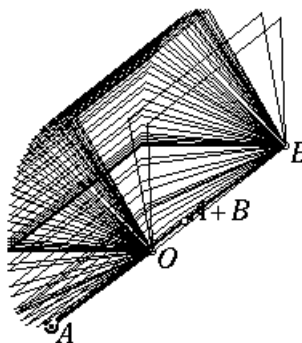
Shrinking parallelograms

Or, think of a continuum of parallelograms:



A shrinking parallelogram

Of course, if A comes in on the other side of B , you get a different image:



A approaching the direction of $-B$

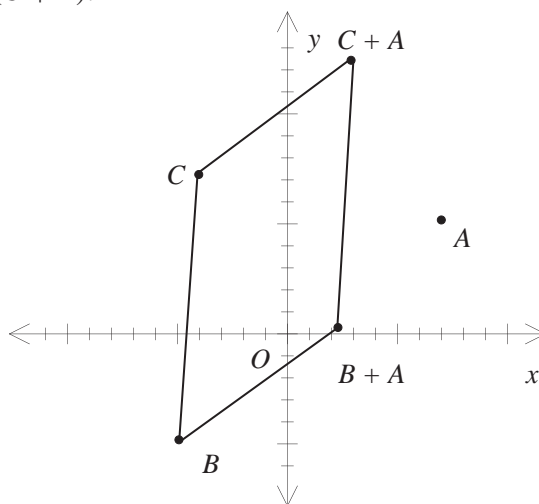
In any case, if O , A , and B are collinear, you can think of \overleftrightarrow{AB} as a one-dimensional space, and $A + B$ can be obtained from A and B in the same way you add coordinates on a one-dimensional number line.

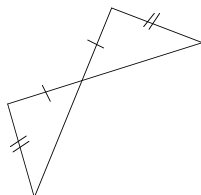
Problem 22 (Student page 106)

$$BC = \sqrt{(c_1 - b_1)^2 + (c_2 - b_2)^2}$$

$$\begin{aligned} (B + A)(C + A) &= \sqrt{[(c_1 + a_1) - (b_1 + a_1)]^2 + [(c_2 + a_2) - (b_2 + a_2)]^2} \\ &= \sqrt{(c_1 - b_1)^2 + (c_2 - b_2)^2} \end{aligned}$$

So $BC = (B + A)(C + A)$.



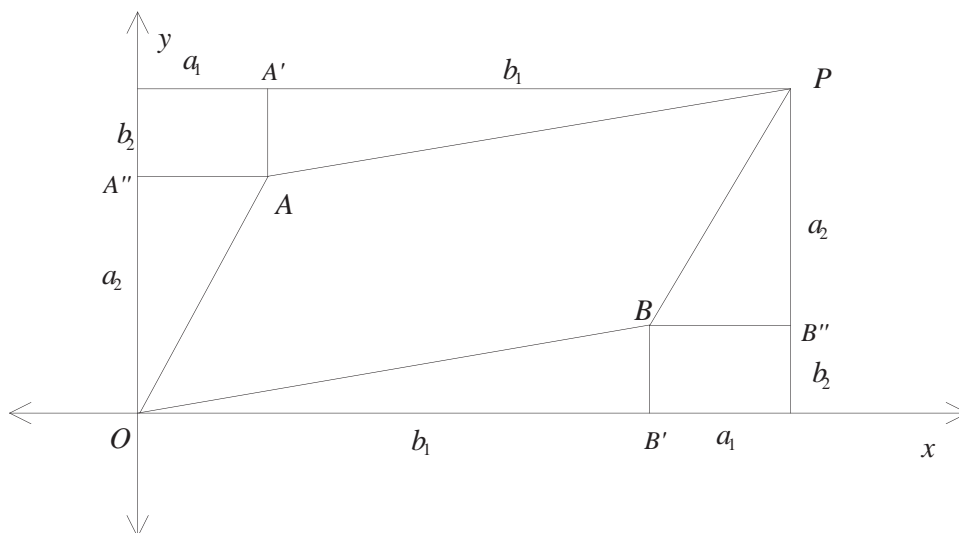


Also, the distance from B to $B + A$ is OA . The distance from C to $C + A$ is OA . So $B(B + A) = C(C + A)$.

Problem 23 (Student page 106) We proved that the opposite sides of $OAPB$ are congruent and declared that that makes it a parallelogram. But that's only true if $OAPB$ is a quadrilateral in the first place. It might “cross itself” and still have $OA = PB$ and $OB = PA$. See the discussion in Investigation 5.17 on page 129 of the Student Module.

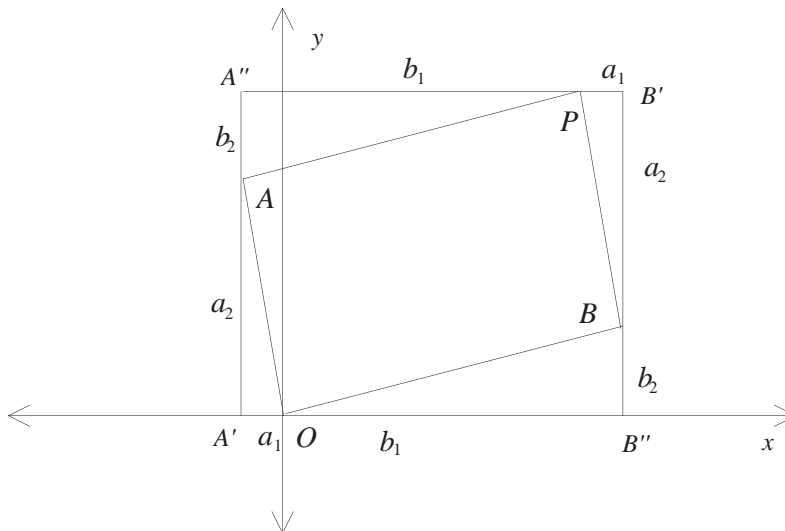
Problem 24 (Student page 107) Note that we will use the previous proof, in which we showed the opposite sides congruent.

- a.** In the figure below, $\triangle PA'A \cong \triangle OB'B$ by HL (hypotenuse-leg test for right triangles). So $\angle A'PA \cong \angle B'OB$. Since $\overline{OB'} \parallel \overline{A'P}$, it must also be true that $\overline{OB} \parallel \overline{AP}$.



Similarly, $\triangle OA''A \cong \triangle PB''B$ by HL. So $\angle A''OA \cong \angle B''PB$. Since $\overline{OA''} \parallel \overline{PB''}$, it must also be true that $\overline{OA} \parallel \overline{BP}$.

- b. In the figure below, $\triangle AOA' \cong \triangle BPB'$ by HL, so $\angle A'AO \cong \angle B'BP$. Since $\overline{B''B'} \parallel \overline{A''A'}$, it must also be true that $\overline{PB} \parallel \overline{OA}$.



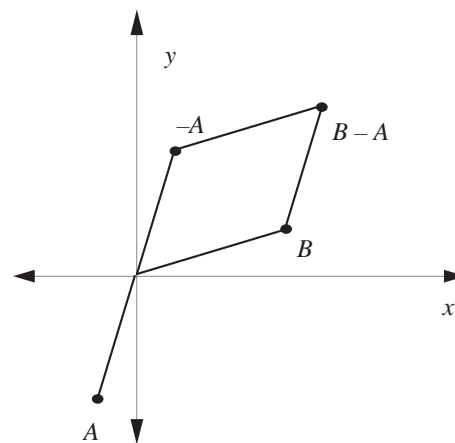
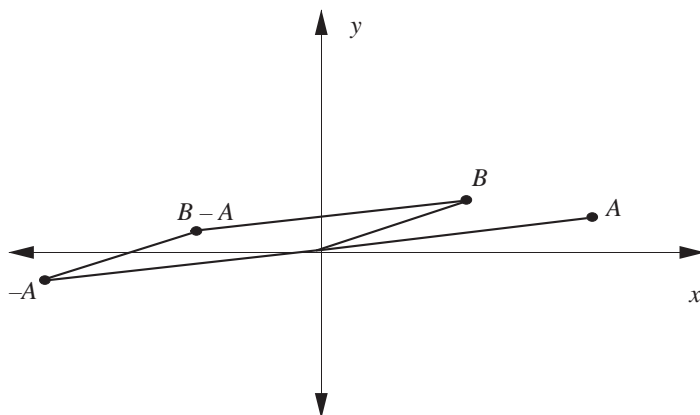
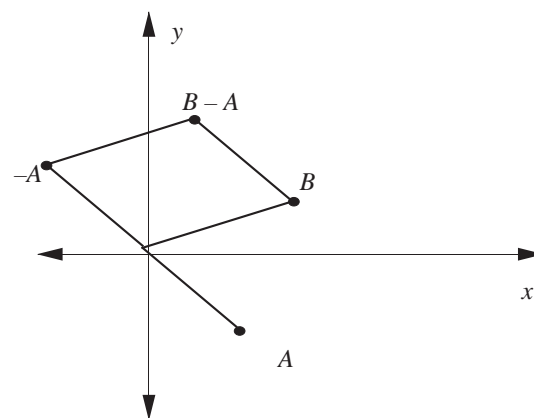
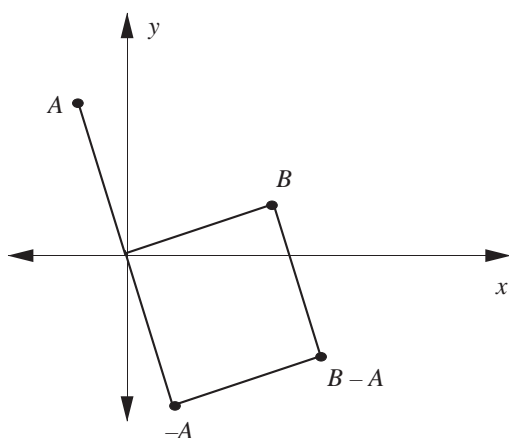
Similarly, $\triangle APA'' \cong \triangle BOB''$ by HL, so $\angle B''OB \cong \angle A''PA$. Since $\overline{A''B'} \parallel \overline{A'B''}$, it must also be that $\overline{AP} \parallel \overline{OB}$.

- c. Because of the symmetry of the proof about A and B , there are really three cases: A and B are in the same quadrant, A and B are in adjacent quadrants, and A and B are in nonadjacent quadrants. We've proved two cases, so there is just one left.

USING THE THEOREMS

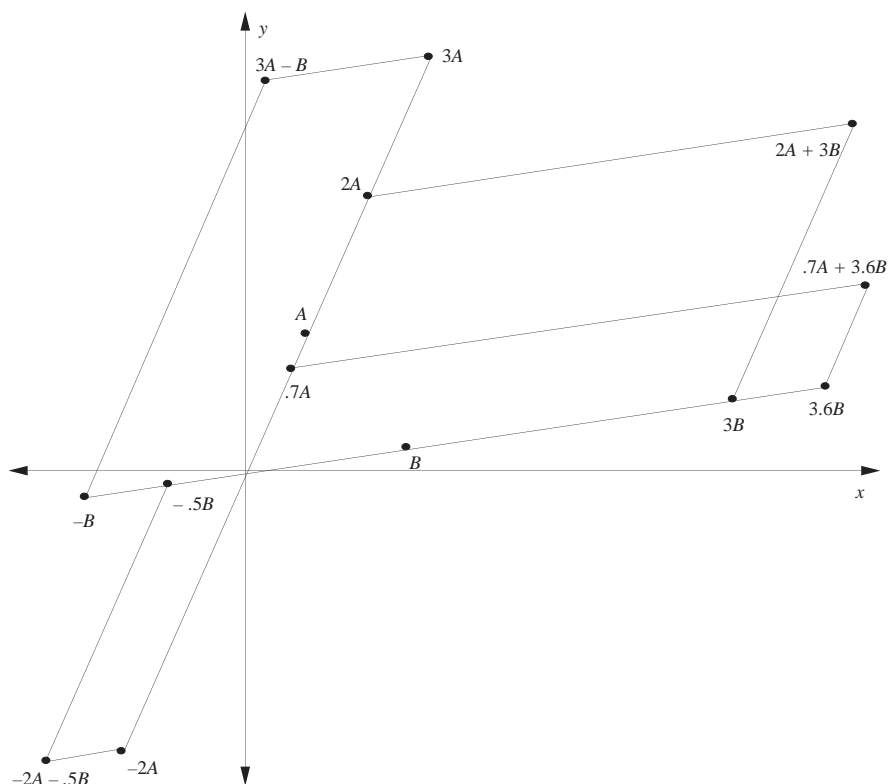
Problem 1 (Student page 108) To locate $-1A$, first draw \overline{OA} . Continue \overline{OA} through the origin in the opposite direction from A , and mark off a length equal to OA in that direction. That new point is $-1A$.

Problem 2 (Student page 108) One way to solve this problem is to first find $-A$ and then to find the sum of $-A$ and B . These pictures demonstrate the method:

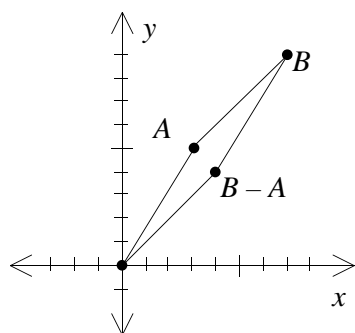
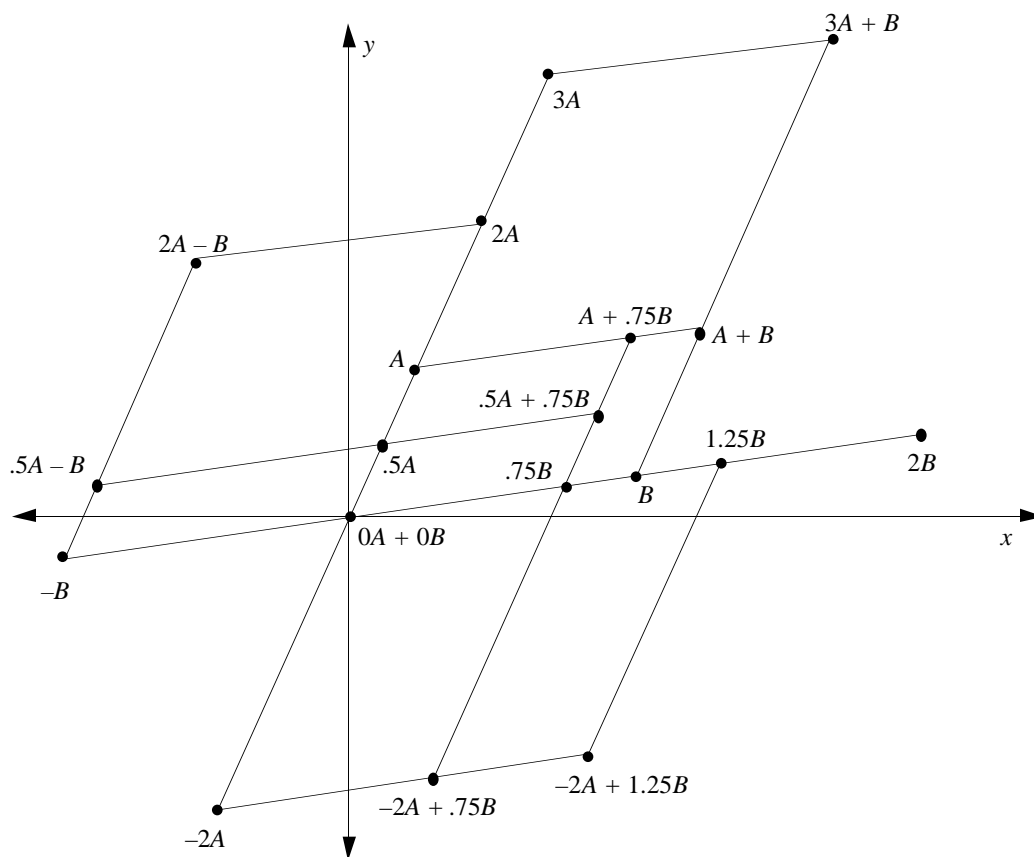


In each of these pictures, draw the segment from O to $B - A$. How does it compare to \overline{AB} ? Can you prove it?

Problem 3 (Student page 109)



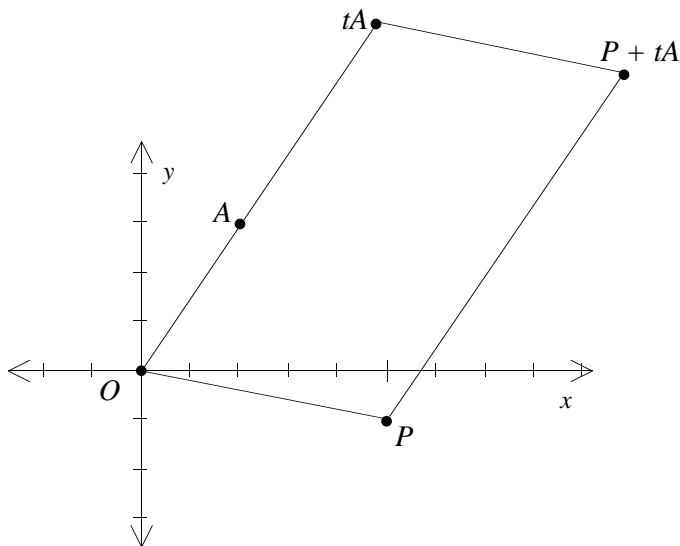
Problem 4 (Student page 109) In the figure below, each point is labeled in the form $x A + y B$. The parallel lines show how we found the values for x and y .



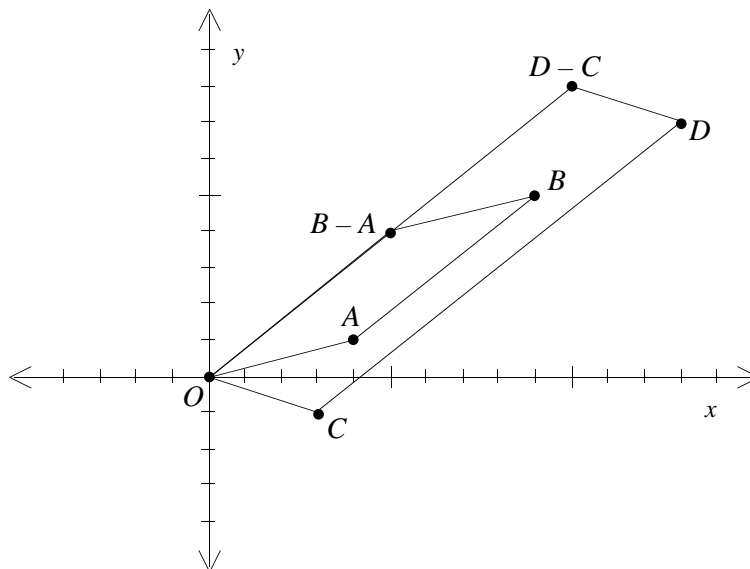
Problem 5 (Student page 110) If $A = (3, 5)$ and $B = (7, 9)$, then $B - A = (4, 4)$. Notice that $(4, 4) + (3, 5) = (7, 9)$. By Theorem 5.4, these three points form a parallelogram with the origin as the fourth vertex. So $\overline{O(B - A)}$ and \overline{AB} are parallel and congruent because they are opposite sides of a parallelogram. (In general: $A + (B - A) = B$, so O , A , B , and $B - A$ form a parallelogram.)

Problem 6 (Student page 110) If we add tA to P , we get the point $P + tA$, so O , P , $P + tA$, and tA form a parallelogram. $\overline{P(P + tA)}$ and $\overline{O(tA)}$ are opposite sides of the parallelogram, so they are parallel and congruent.

Since A and tA are on the same line through the origin, \overline{OA} is parallel to $\overline{P(P + tA)}$ as well.



Problems 7–8 (Student pages 110–111) Since $A = (4, 1)$ and $B = (9, 5)$, $B - A = (5, 4)$. Since $C = (3, -1)$ and $D = (13, 7)$, $D - C = (10, 8)$.



By Theorem 5.5, $\overline{DC} \parallel \overline{O(D - C)}$ and $\overline{AB} \parallel \overline{O(B - A)}$. Note that $D - C = 2(B - A)$, so $D - C$ is on the same line through the origin as $B - A$. Therefore, \overline{AB} and \overline{CD} are parallel to the same line through the origin, so they must be parallel to each other. Thus, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

In general, if $B - A = t(D - C)$, then $B - A$ is on the same line through the origin as $D - C$. From Theorem 5.5, we know that $\overline{DC} \parallel \overline{O(D - C)}$ and also that $\overline{AB} \parallel \overline{O(B - A)}$. Thus, \overline{AB} and \overline{CD} are parallel to the same line through the origin, and must therefore be parallel to each other. Thus, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Problem 9 (Student page 111) First we will use the distance formula to show that if $M = \frac{1}{2}(A + B)$, then $BM = AM$. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. Then

$$\begin{aligned} M &= \frac{1}{2}(a_1 + b_1, a_2 + b_2) \\ &= \left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2} \right). \end{aligned}$$

By the distance formula,

$$\begin{aligned} BM &= \sqrt{\left[b_1 - \left(\frac{a_1 + b_1}{2} \right) \right]^2 + \left[b_2 - \left(\frac{a_2 + b_2}{2} \right) \right]^2} \\ &= \sqrt{\left(\frac{b_1 - a_1}{2} \right)^2 + \left(\frac{b_2 - a_2}{2} \right)^2} \end{aligned}$$

and

$$\begin{aligned} AM &= \sqrt{\left[a_1 - \frac{a_1 + b_1}{2} \right]^2 + \left[a_2 - \frac{a_2 + b_2}{2} \right]^2} \\ &= \sqrt{\left(\frac{a_1 - b_1}{2} \right)^2 + \left(\frac{a_2 - b_2}{2} \right)^2}. \end{aligned}$$

Since

$$\left(\frac{b_1 - a_1}{2} \right)^2 = \left(\frac{a_1 - b_1}{2} \right)^2$$

and

$$\left(\frac{b_2 - a_2}{2} \right)^2 = \left(\frac{a_2 - b_2}{2} \right)^2,$$

we have shown that

$$BM = AM.$$

It remains only to show that M is on the same line as \overline{AB} .

$$\begin{aligned} B - M &= B - \frac{1}{2}(A + B) \\ &= \frac{1}{2}B - \frac{1}{2}A \\ &= \frac{1}{2}(B - A) \end{aligned}$$

So $B - M = t(B - A)$ (in this case $t = \frac{1}{2}$), so we know from Problem 8 that $\overleftrightarrow{AB} \parallel \overleftrightarrow{MB}$. But two parallel lines that share a point must be the same line, so M is on \overleftrightarrow{AB} , and is therefore the midpoint of \overline{AB} .

Are all these algebraic moves justified?

Problem 10 (Student page 111) Since A is the midpoint of \overline{PX} , we can write

$$\begin{aligned} \frac{1}{2}(P + X) &= A \\ \frac{1}{2}X &= A - \frac{1}{2}P \\ X &= 2A - P. \end{aligned}$$

Draw a picture here.

Problem 11 (Student page 111) Since O is the midpoint of \overline{AC} , we can write

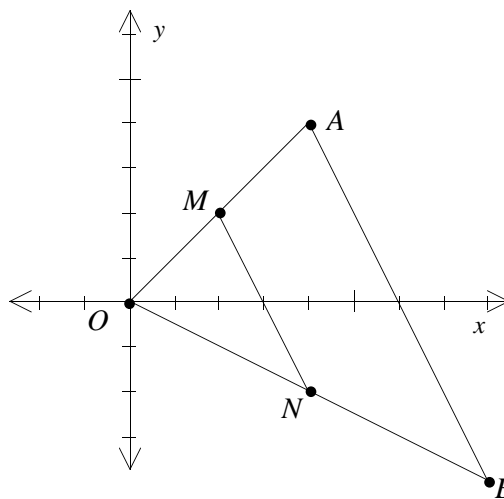
$$\begin{aligned} \frac{1}{2}(A + C) &= O \\ \frac{1}{2}C &= O - \frac{1}{2}A \\ C &= 2O - A \\ &= -A. \end{aligned}$$

Problem 12 (Student page 111) Remember that $A = (4, 4)$ and $B = (8, -4)$, so we can find the coordinates for M and N :

$$\begin{aligned} M &= \frac{1}{2}(O + A) \\ &= \frac{1}{2}A \\ &= (2, 2) \end{aligned}$$

and

$$\begin{aligned} N &= \frac{1}{2}(O + B) \\ &= \frac{1}{2}B \\ &= (4, -2). \end{aligned}$$



To test that \overline{AB} and \overline{MN} are parallel, we will check that we can write $M - N = t(A - B)$ for some number t :

$$\begin{aligned} A - B &= (-4, 8) \\ M - N &= (-2, 4) \\ &= \frac{1}{2}(A - B). \end{aligned}$$

Can you use the fact that $M - N = \frac{1}{2}(A - B)$ to avoid the distance formula?

To check that $MN = \frac{1}{2}AB$, we will use the distance formula:

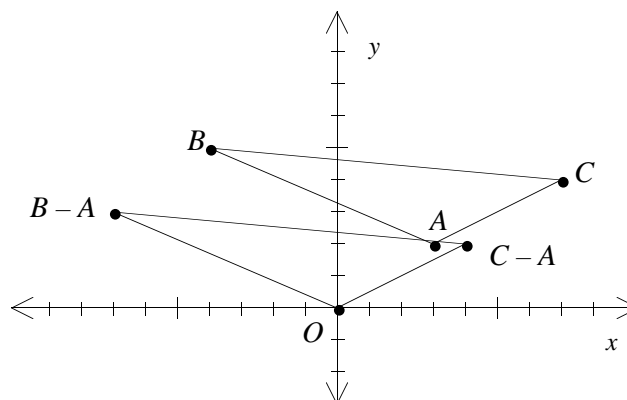
$$\begin{aligned} AB &= \sqrt{(8 - 4)^2 + (-4 - 4)^2} \\ &= \sqrt{16 + 64} \\ &= 4\sqrt{5} \end{aligned}$$

and

$$\begin{aligned}
 MN &= \sqrt{(4-2)^2 + (-2-2)^2} \\
 &= \sqrt{4+16} \\
 &= 2\sqrt{5} \\
 &= \frac{1}{2}AB.
 \end{aligned}$$

It makes sense that the two triangles will be congruent; we've just translated each of the three vertices by $-A$.

Problems 13–14 (Student page 111) We'll prove the two triangles are congruent by SSS.



By Theorem 5.5, we know that $\overline{AB} \cong \overline{O(B-A)}$. We can check with the distance formula:

$$\begin{aligned}
 AB &= \sqrt{7^2 + 3^2} \\
 &= \sqrt{58}
 \end{aligned}$$

$$\begin{aligned}
 O(B-A) &= \sqrt{(-4-3)^2 + (5-2)^2} \\
 &= \sqrt{58}.
 \end{aligned}$$

Theorem 5.5 also tells us that $\overline{AC} \cong \overline{O(C-A)}$. We can again check with the distance formula:

$$\begin{aligned}
 AC &= \sqrt{4^2 + 2^2} \\
 &= \sqrt{20}
 \end{aligned}$$

and

$$\begin{aligned} O(C - A) &= \sqrt{(7 - 3)^2 + (4 - 2)^2} \\ &= \sqrt{20}. \end{aligned}$$

Now we have only to check that $\overline{BC} \cong \overline{(B - A)(C - A)}$. We will use the distance formula one more time:

$$\begin{aligned} BC &= \sqrt{(7 + 4)^2 + (4 - 5)^2} \\ &= \sqrt{122} \end{aligned}$$

and

$$\begin{aligned} (C - A)(B - A) &= \sqrt{(-7 - 4)^2 + (3 - 2)^2} \\ &= \sqrt{122}. \end{aligned}$$

So, by SSS, we have shown that

$$\triangle ABC \cong \triangle O(B - A)(C - A).$$

In the general case, we just use variables in the distance formulas; the calculations are exactly the same. We'll let $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$. This means that $B - A = (b_1 - a_1, b_2 - a_2)$ and $C - A = (c_1 - a_1, c_2 - a_2)$.

$$\begin{aligned} AB &= \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2} \\ O(B - A) &= \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2} \\ &= AB \end{aligned}$$

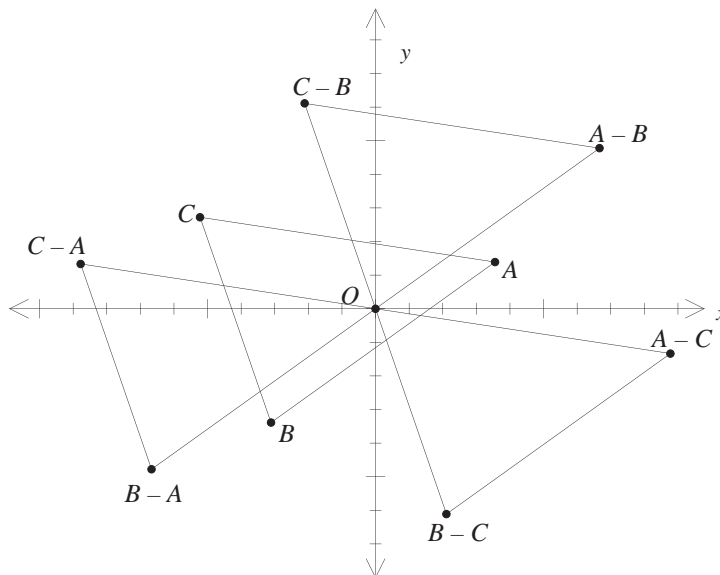
$$\begin{aligned} AC &= \sqrt{(c_1 - a_1)^2 + (c_2 - a_2)^2} \\ O(C - A) &= \sqrt{(c_1 - a_1)^2 + (c_2 - a_2)^2} \\ &= AC \end{aligned}$$

$$BC = \sqrt{(c_1 - b_1)^2 + (c_2 - b_2)^2}$$

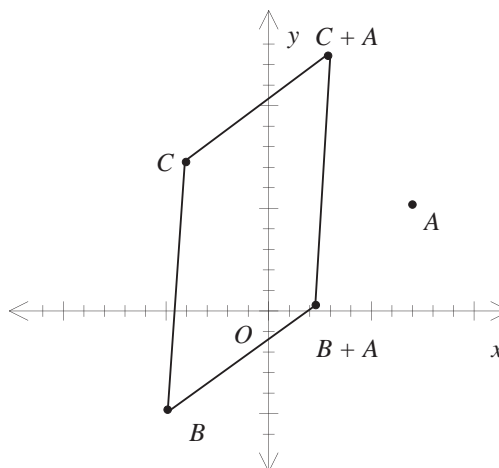
$$\begin{aligned} (B - A)(C - A) &= \sqrt{[(c_1 - a_1) - (b_1 - a_1)]^2 + [(c_2 - a_2) - (b_2 - a_2)]^2} \\ &= \sqrt{(c_1 - b_1)^2 + (c_2 - b_2)^2} \\ &= BC \end{aligned}$$

So we have shown that $\overline{BC} \cong \overline{(B - A)(C - A)}$, $\overline{AB} \cong \overline{O(B - A)}$, and $\overline{AC} \cong \overline{O(C - A)}$. By SSS, we have $\triangle ABC \cong \triangle O(B - A)(C - A)$.

Problem 15 (Student page 112) Here is one picture. Answers will vary slightly depending on initial choice of A , B , and C . In all pictures, the four triangles should be congruent, and three of them should have a vertex at the origin. All four should have the same orientation (their sides should be parallel).



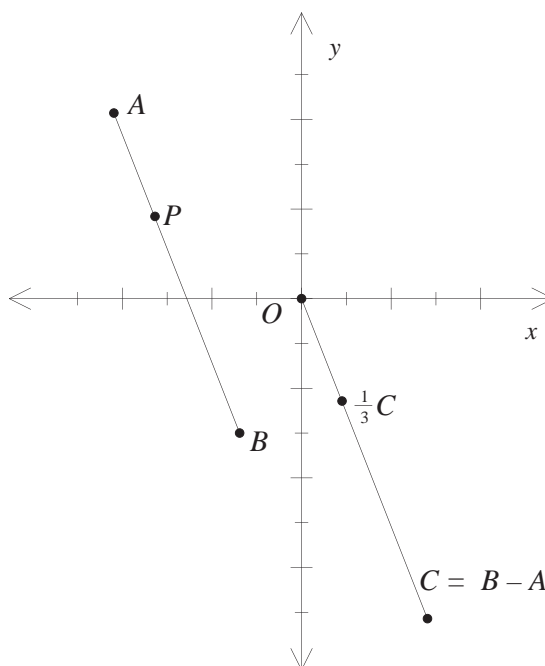
Problem 16 (Student page 112) In Problem 22 of Investigation 5.13, we showed that opposite sides of this figure were congruent. Now we can show that they are in fact parallel (so we are sure that the figure is not a bow tie), provided that we accept Theorem 5.4.



To show that opposite sides are parallel, notice that $(B + A) - (C + A) = B - C$. So $(B + A) - (C + A) = t(B - C)$ (where $t = 1$), and $\overline{(B + A)(C + A)} \parallel \overline{BC}$.

Similarly, $(B + A) - B = A$ and $(C + A) - C = A$. So $(B + A) - B = t((C + A) - C)$, and $\overline{B(B + A)} \parallel \overline{C(C + A)}$.

Problem 17 (Student page 112) The point $\frac{1}{3}(A + A + B)$ is one third of the way from A to B . Here's one way to think about it:



There is some point C such that $A + C = B$. We can solve this equation for C : $C = B - A$. Now, $A + \frac{1}{3}C$ is the point we want since

$$\begin{aligned} P &= A + \frac{1}{3}C \\ &= A + \frac{1}{3}(B - A) \\ &= \frac{2}{3}A + \frac{1}{3}B. \end{aligned}$$

THE ALGEBRA OF POINTS

Problem 1 (Student page 113) Note that we use properties of *numbers* (the coordinates of the points and the numbers d and e) to prove similar properties for points.

1.

$$\begin{aligned}
 A + B &= (a_1, a_2) + (b_1, b_2) \\
 &= (a_1 + b_1, a_2 + b_2) \\
 &= (b_1 + a_1, b_2 + a_2) \\
 &= (b_1, b_2) + (a_1, a_2) \\
 &= B + A
 \end{aligned}$$

2.

$$\begin{aligned}
 A + (B + C) &= (a_1, a_2) + [(b_1, b_2) + (c_1, c_2)] \\
 &= (a_1, a_2) + (b_1 + c_1, b_2 + c_2) \\
 &= (a_1 + (b_1 + c_1), a_2 + (b_2 + c_2)) \\
 &= ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2) \\
 &= (a_1 + b_1, a_2 + b_2) + (c_1, c_2) \\
 &= [(a_1, a_2) + (b_1, b_2)] + (c_1, c_2) \\
 &= (A + B) + C
 \end{aligned}$$

3.

$$\begin{aligned}
 A + O &= (a_1, a_2) + (0, 0) \\
 &= (a_1 + 0, a_2 + 0) \\
 &= (a_1, a_2) \\
 &= A
 \end{aligned}$$

4.

$$\begin{aligned}
 A + (-1A) &= (a_1, a_2) + [-1(a_1, a_2)] \\
 &= (a_1, a_2) + (-a_1, -a_2) \\
 &= (a_1 - a_1, a_2 - a_2) \\
 &= (0, 0) \\
 &= O
 \end{aligned}$$

5.

$$\begin{aligned}
 (d + e)A &= (d + e)(a_1, a_2) \\
 &= ((d + e)a_1, (d + e)a_2) \\
 &= (da_1 + ea_1, da_2 + ea_2) \\
 &= (da_1, da_2) + (ea_1, ea_2) \\
 &= d(a_1, a_2) + e(a_1, a_2) \\
 &= dA + eA
 \end{aligned}$$

6.

$$\begin{aligned}
 d(A + B) &= d((a_1, a_2) + (b_1, b_2)) \\
 &= d(a_1 + b_1, a_2 + b_2) \\
 &= (d(a_1 + b_1), d(a_2 + b_2)) \\
 &= (da_1 + db_1, da_2 + db_2) \\
 &= (da_1, da_2) + (db_1, db_2) \\
 &= d(a_1, a_2) + d(b_1, b_2) \\
 &= dA + dB
 \end{aligned}$$

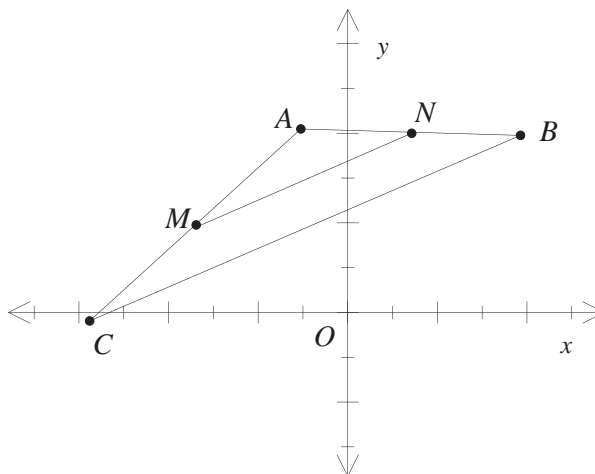
7.

$$\begin{aligned}
 d(eA) &= d(e(a_1, a_2)) \\
 &= d(ea_1, ea_2) \\
 &= (d(ea_1), d(ea_2)) \\
 &= ((de)a_1, (de)a_2) \\
 &= de(a_1, a_2) \\
 &= (de)A
 \end{aligned}$$

8.

$$\begin{aligned}
 1A &= 1(a_1, a_2) \\
 &= (1a_1, 1a_2) \\
 &= (a_1, a_2) \\
 &= A
 \end{aligned}$$

Problem 2 (Student page 113) We are now proving the general case of the Midline Theorem (instead of the specific case we checked in Problem 12 of Investigation 5.14).



Consider some $\triangle ABC$ where $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$. We'll let M be the midpoint of \overline{AC} , so $M = \frac{1}{2}(A + C)$. Likewise, N will be the midpoint of \overline{AB} , so $N = \frac{1}{2}(A + B)$. We can find the coordinates for M and N :

$$M = \frac{1}{2}(a_1 + c_1, a_2 + c_2) = \left(\frac{a_1 + c_1}{2}, \frac{a_2 + c_2}{2} \right)$$

$$N = \frac{1}{2}(a_1 + b_1, a_2 + b_2) = \left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2} \right).$$

To check whether \overline{MN} is parallel to \overline{CB} , we need to show that $M - N = t(C - B)$ for some real number t :

$$\begin{aligned} M - N &= \frac{1}{2}(A + C) - \frac{1}{2}(A + B) \\ &= \frac{1}{2}A + \frac{1}{2}C - \frac{1}{2}A - \frac{1}{2}B \\ &= \frac{1}{2}(C - B). \end{aligned}$$

So $M - N = \frac{1}{2}(C - B)$, and we conclude that $\overline{MN} \parallel \overline{CB}$.

Can you see where we use results from Theorem 5.6 in these calculations?

Now we will use the distance formula to check that $MN = \frac{1}{2}BC$:

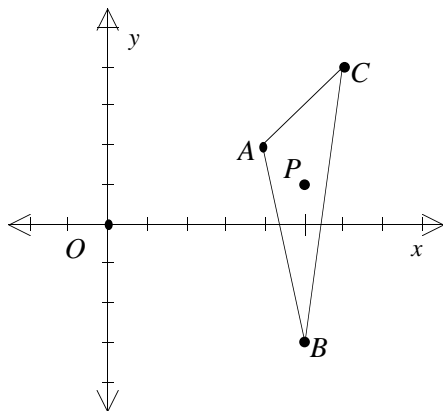
$$\begin{aligned}
 BC &= \sqrt{(c_1 - b_1)^2 + (c_2 - b_2)^2} \\
 MN &= \sqrt{\left(\frac{a_1 + c_1}{2} - \frac{a_1 + b_1}{2}\right)^2 + \left(\frac{a_2 + c_2}{2} - \frac{a_2 + b_2}{2}\right)^2} \\
 &= \sqrt{\left(\frac{c_1 - b_1}{2}\right)^2 + \left(\frac{c_2 - b_2}{2}\right)^2} \\
 &= \sqrt{\left(\frac{1}{2}\right)^2 [(c_1 - b_1)^2 + (c_2 - b_2)^2]} \\
 &= \frac{1}{2} \sqrt{(c_1 - b_1)^2 + (c_2 - b_2)^2} \\
 &= \frac{1}{2} BC.
 \end{aligned}$$

Problem 3 (Student page 113)

- $\frac{1}{3}A + \frac{2}{3}B$ is $\frac{2}{3}$ of the way from A to B .
- $\frac{2}{3}A + \frac{1}{3}B$ is $\frac{1}{3}$ of the way from A to B .
- $\frac{1}{4}A + \frac{3}{4}B$ is $\frac{3}{4}$ of the way from A to B .
- $\frac{3}{4}A + \frac{1}{4}B$ is $\frac{1}{4}$ of the way from A to B .
- $\frac{3}{5}A + \frac{2}{5}B$ is $\frac{2}{5}$ of the way from A to B .
- $kA + (1 - k)B$, (here, $0 \leq k \leq 1$) is $(1 - k)$ of the way from A to B .

Problem 4 (Student page 114) Using the given coordinates $A = (4, 2)$, $B = (5, -3)$, and $C = (6, 4)$, we can calculate the coordinates of P :

$$\begin{aligned}
 P &= \frac{1}{3}(A + B + C) \\
 &= \frac{1}{3}(15, 3) \\
 &= (5, 1).
 \end{aligned}$$



Now we will calculate the coordinates of each midpoint:

$$\begin{aligned} M_{\overline{AB}} &= \frac{1}{2}(A + B) \\ &= \frac{1}{2}(9, -1) \\ &= \left(\frac{9}{2}, -\frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned} M_{\overline{AC}} &= \frac{1}{2}(A + C) \\ &= \frac{1}{2}(10, 6) \\ &= (5, 3) \end{aligned}$$

$$\begin{aligned} M_{\overline{BC}} &= \frac{1}{2}(B + C) \\ &= \frac{1}{2}(11, 1) \\ &= \left(\frac{11}{2}, \frac{1}{2}\right) \end{aligned}$$

The point $\frac{2}{3}$ of the way from A to $M_{\overline{BC}}$:

$$\begin{aligned} P' &= \frac{1}{3}(4, 2) + \frac{2}{3}\left(\frac{11}{2}, \frac{1}{2}\right) \\ &= \left(\frac{4}{3}, \frac{2}{3}\right) + \left(\frac{11}{3}, \frac{1}{3}\right) \\ &= (5, 1) \\ &= P \end{aligned}$$

The point $\frac{2}{3}$ of the way from B to $M_{\overline{AC}}$:

$$\begin{aligned} P' &= \frac{1}{3}(5, -3) + \frac{2}{3}(5, 3) \\ &= \left(\frac{5}{3}, -1\right) + \left(\frac{10}{3}, 2\right) \\ &= (5, 1) \\ &= P \end{aligned}$$

The point $\frac{2}{3}$ of the way from C to $M_{\overline{AB}}$:

$$\begin{aligned} P' &= \frac{1}{3}(6, 4) + \frac{2}{3}\left(\frac{9}{2}, -\frac{1}{2}\right) \\ &= \left(2, \frac{4}{3}\right) + \left(3, -\frac{1}{3}\right) \\ &= (5, 1) \\ &= P \end{aligned}$$

So P is in fact the point $\frac{2}{3}$ of the way from any vertex to the midpoint of the opposite side.

Problem 5 (Student page 114) This is the generalization of Problem 4. Here, we do the calculations with points instead of with coordinates. We'll use the following:

$$\begin{aligned} P &= \frac{1}{3}(A + B + C) & M_{\overline{AB}} &= \frac{1}{2}(A + B) \\ M_{\overline{BC}} &= \frac{1}{2}(B + C) & M_{\overline{AC}} &= \frac{1}{2}(A + C). \end{aligned}$$

We want to show that

$$\begin{aligned} P &= \frac{1}{3}A + \frac{2}{3}M_{\overline{BC}} \\ &= \frac{1}{3}A + \frac{2}{3}M_{\overline{AC}} \\ &= \frac{1}{3}A + \frac{2}{3}M_{\overline{AB}}. \end{aligned}$$

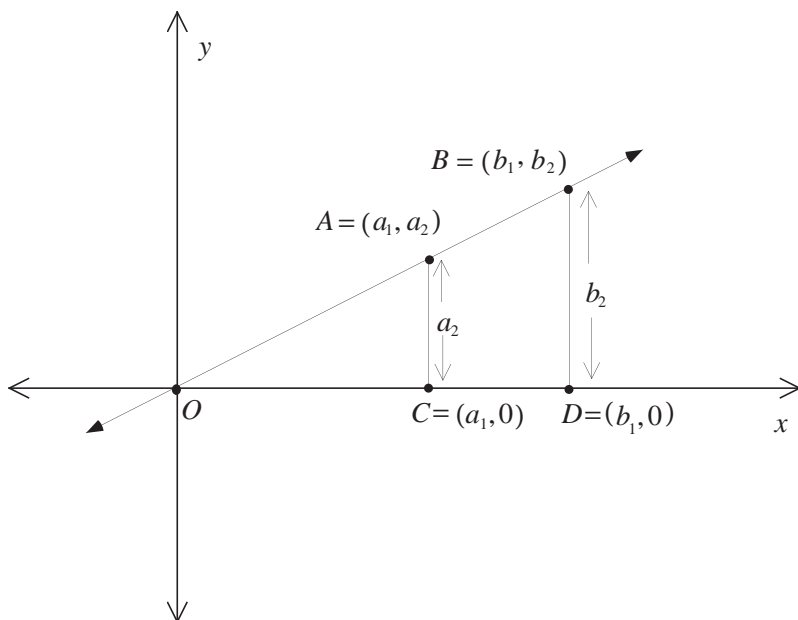
$$\begin{aligned} \frac{1}{3}A + \frac{2}{3}M_{\overline{BC}} &= \frac{1}{3}A + \frac{2}{3}\left[\frac{1}{2}(B + C)\right] \\ &= \frac{1}{3}A + \frac{1}{3}(B + C) \\ &= \frac{1}{3}(A + B + C) \\ &= P \end{aligned}$$

$$\begin{aligned}\frac{1}{3}B + \frac{2}{3}M_{\overline{AC}} &= \frac{1}{3}B + \frac{2}{3}\left[\frac{1}{2}(A + C)\right] \\ &= \frac{1}{3}B + \frac{1}{3}(A + C) \\ &= \frac{1}{3}(A + B + C) \\ &= P\end{aligned}$$

$$\begin{aligned}\frac{1}{3}C + \frac{2}{3}M_{\overline{AB}} &= \frac{1}{3}C + \frac{2}{3}\left[\frac{1}{2}(A + B)\right] \\ &= \frac{1}{3}C + \frac{1}{3}(A + B) \\ &= \frac{1}{3}(A + B + C) \\ &= P\end{aligned}$$

MORE ON SCALING POINTS

Problem 1 (Student page 117) Here's one version of the proof with more details than are provided in the Student Module:



We know that $A = (a_1, a_2)$, $B = (b_1, b_2)$, B is on the line \overleftrightarrow{OA} , and A is between B and O . In the picture, we have dropped perpendiculars from A and B to the x -axis. These lines intersect the axis at $C = (a_1, 0)$ and $D = (b_1, 0)$.

We can conclude that $\triangle BDO \sim \triangle ACO$. Why? $\overline{AC} \parallel \overline{BD}$ because both are perpendicular to the x -axis. Therefore, \overleftrightarrow{AC} and \overleftrightarrow{BD} are parallel lines cut by two transversals, \overleftrightarrow{OB} and \overleftrightarrow{OD} . When parallel lines are cut by a transversal, corresponding angles are congruent, so $\angle ODB \cong \angle OCA$ and $\angle OBD \cong \angle OAC$. We also have $\angle BOD \cong \angle AOC$, since these are two names for the same angle. We conclude that $\triangle BDO \sim \triangle ACO$ by the AAA Similarity Theorem.

Since the triangles are similar, their sides are proportional. Thus,

$$\frac{BD}{AC} = \frac{OD}{OC} = \frac{OB}{OA}.$$

Let's call this ratio k . Note that $AC = a_2$, $OC = a_1$, $BD = b_2$, and $OD = b_1$. Substituting into the above, we get

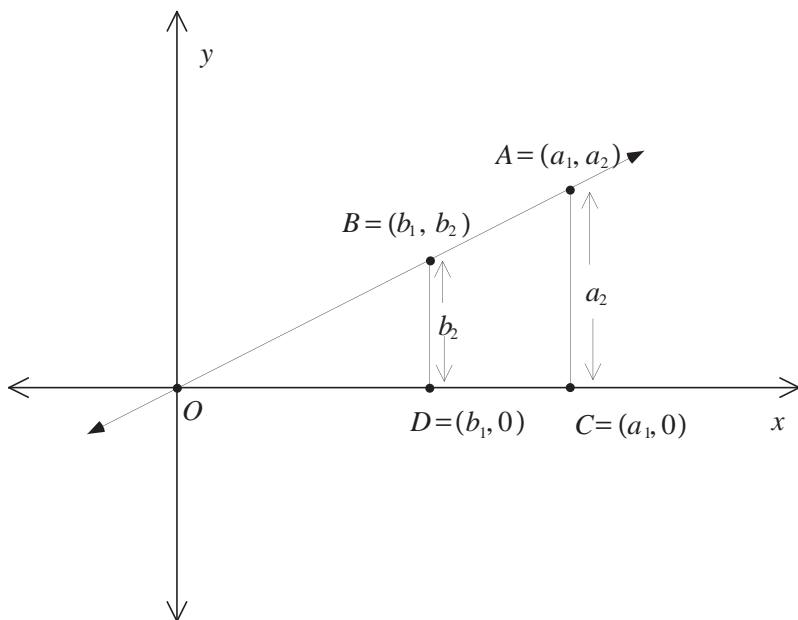
$$\frac{b_1}{a_1} = k \quad b_1 = ka_1$$

and

$$\frac{b_2}{a_2} = k \quad b_2 = ka_2.$$

So $B = (b_1, b_2) = (ka_1, ka_2) = k(a_1, a_2) = kA$, and we have shown that B is a multiple of A .

Problem 2 (Student page 117) The only thing that has changed from Problem 1 is that B is now between O and A .



Again, we drop perpendiculars from A and B to the x -axis, calling the intersections $C = (a_1, 0)$ and $D = (b_1, 0)$.

$\triangle BDO \sim \triangle ACO$, exactly as before. Why? As in the figure for Problem 1, $\overline{BD} \parallel \overline{AC}$. Again we get congruent corresponding angles and AAA similarity. Now we can write the same proportions:

$$\frac{BD}{AC} = \frac{OD}{OC} = \frac{OB}{OA}.$$

This is a different k from the problem above. In one case we have $k > 1$ and in the other we have $0 < k < 1$. Which is which?

Let's call this ratio k . Note that $AC = a_2$, $OC = a_1$, $BD = b_2$, and $OD = b_1$. Substituting into the above, we get

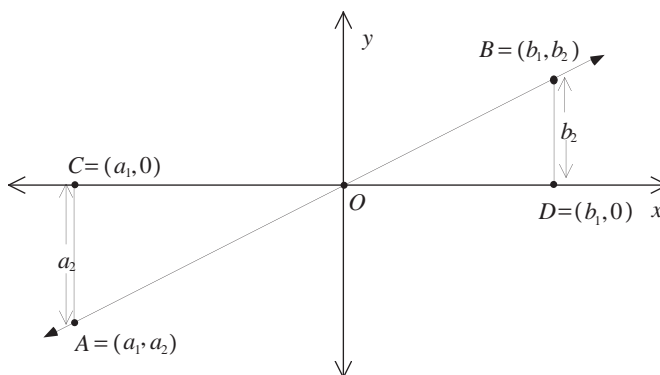
$$\frac{b_1}{a_1} = k \quad b_1 = ka_1$$

and

$$\frac{b_2}{a_2} = k \quad b_2 = ka_2.$$

So $B = (b_1, b_2) = (ka_1, ka_2) = k(a_1, a_2) = kA$, and we have shown that B is a multiple of A .

Problem 3 (Student page 117) Now O is between A and B . Compare the following picture to those shown in the solutions for Problems 1 and 2.



The proof follows in much the same way, though. Again we drop perpendiculars to the x -axis and show that $\triangle ACO \sim \triangle BDO$. This time we have $\angle ACO \cong \angle BDO$ because they are both right angles. We also have $\angle AOC \cong \angle BOD$ because they are vertical angles. So we have AA similarity and proportional sides:

$$\frac{BD}{AC} = \frac{OD}{OC} = \frac{OB}{OA}.$$

Let's call this ratio k . Note that $AC = a_2$, $OC = a_1$, $BD = b_2$, and $OD = b_1$. Substituting into the above, we get

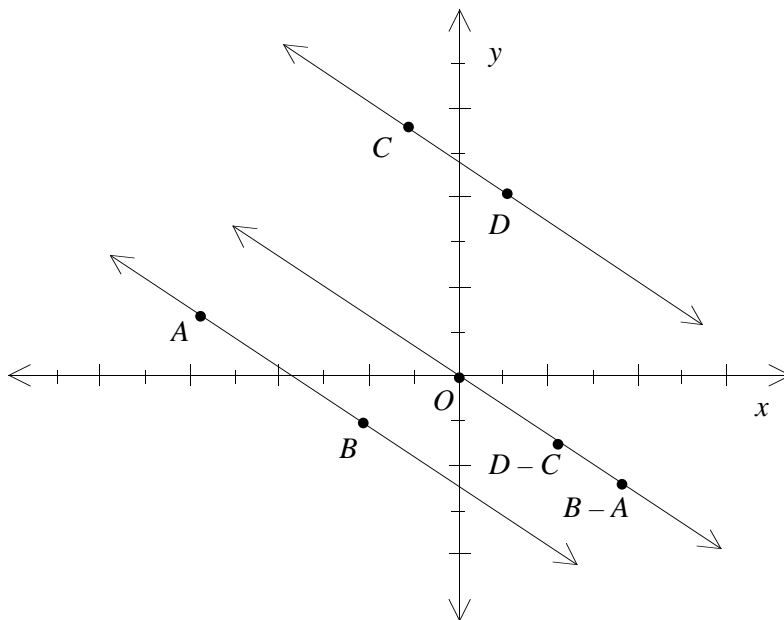
$$\frac{b_1}{a_1} = k \quad b_1 = ka_1$$

and

$$\frac{b_2}{a_2} = k \quad b_2 = ka_2.$$

So $B = (b_1, b_2) = (ka_1, ka_2) = k(a_1, a_2) = kA$, and we have shown that B is a multiple of A .

Problem 4 (Student page 117) Remember also that $A, B, B - A$, and O form a parallelogram with $\overrightarrow{AB} \parallel \overrightarrow{O(B - A)}$. Similarly, $C, D, D - C$, and O form a parallelogram with $\overrightarrow{CD} \parallel \overrightarrow{O(D - C)}$.



Since $\overrightarrow{AB} \parallel \overrightarrow{CD}$, we know that $\overrightarrow{AB} \parallel \overrightarrow{O(D - C)}$ and therefore that $\overrightarrow{O(B - A)} \parallel \overrightarrow{O(D - C)}$. These are two parallel lines through the origin, so they must coincide. That is, $(B - A)$ and $(D - C)$ must be on the same line through the origin. By Theorem 5.7, we see that $(B - A) = k(D - C)$.

Problem 5 (Student page 120)

- a. If $c \geq 1$, then $c - 1 \geq 0$, so $\sqrt{(c - 1)^2} = (c - 1)$. For all c , it's true that $\sqrt{(c - 1)^2} = |c - 1|$.
- b. If $c \geq 0$, then $\sqrt{c^2} = c$. For all c , it's true that $\sqrt{c^2} = |c|$.

Problem 6 (Student page 121)

- a.** If $c \geq 1$, then $|c - 1| = (c - 1)$ and $|c| = c$.

$$OA = \sqrt{a_1^2 + a_2^2}$$

$$AB = (c - 1)\sqrt{a_1^2 + a_2^2}$$

$$OB = c\sqrt{a_1^2 + a_2^2}$$

$$\begin{aligned} OA + AB &= \sqrt{a_1^2 + a_2^2} + (c - 1)\sqrt{a_1^2 + a_2^2} \\ &= c\sqrt{a_1^2 + a_2^2} \\ &= OB \end{aligned}$$

- b.** If $0 \leq c \leq 1$, then $|c - 1| = (1 - c)$ and $|c| = c$.

$$OA = \sqrt{a_1^2 + a_2^2}$$

$$AB = (1 - c)\sqrt{a_1^2 + a_2^2}$$

$$OB = c\sqrt{a_1^2 + a_2^2}$$

$$\begin{aligned} OB + AB &= c\sqrt{a_1^2 + a_2^2} + (1 - c)\sqrt{a_1^2 + a_2^2} \\ &= \sqrt{a_1^2 + a_2^2} \\ &= OA \end{aligned}$$

- c.** If $c \leq 0$, then $|c - 1| = (1 - c)$ and $|c| = -c$.

$$OA = \sqrt{a_1^2 + a_2^2}$$

$$AB = (1 - c)\sqrt{a_1^2 + a_2^2}$$

$$OB = -c\sqrt{a_1^2 + a_2^2}$$

$$\begin{aligned} OB + OA &= -c\sqrt{a_1^2 + a_2^2} + \sqrt{a_1^2 + a_2^2} \\ &= (1 - c)\sqrt{a_1^2 + a_2^2} \\ &= AB \end{aligned}$$

Problem 7 (Student page 121) $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, and A is collinear with B and the origin.

We can drop perpendiculars from A and B to the xy -plane, intersecting it at $C = (a_1, a_2, 0)$ and $D = (b_1, b_2, 0)$. First notice that D will be collinear with C and the origin. Why? Well, $\overrightarrow{AC} \parallel \overrightarrow{BD}$ (since they are both perpendicular to the same plane). Also O , A , and B are collinear, so $\triangle AOC \sim \triangle BOD$, and O , C , and D must also be

collinear. Then (from the two-dimensional version of our theorem) $D = kC$, which we can also write as $(b_1, b_2, 0) = (ka_1, ka_2, 0)$.

Because of the similar triangles, we also have proportionality:

$$\frac{OB}{OA} = \frac{OD}{OC} = \frac{BD}{AC}.$$

Since $D = kC$, the other ratios here must also be k . Note that $AC = a_3$ and $BD = b_3$, so $\frac{b_3}{a_3} = k$ and $b_3 = ka_3$.

We can now write $B = (b_1, b_2, b_3) = (ka_1, ka_2, ka_3) = k(a_1, a_2, a_3) = kA$, and we have proved the three-dimensional version of Theorem 5.7.

THEOREM

\mathbb{R}^3 means
(three-dimensional)
Cartesian space.

If A and B are points in \mathbb{R}^3 so that $B = cA$, the distance formulas look like this:

$$\begin{aligned} OA &= \sqrt{a_1^2 + a_2^2 + a_3^2} \\ OB &= \sqrt{(ca_1)^2 + (ca_2)^2 + (ca_3)^2} \\ &= \sqrt{c^2a_1^2 + c^2a_2^2 + c^2a_3^2} \\ &= \sqrt{c^2(a_1^2 + a_2^2 + a_3^2)} \\ &= |c|\sqrt{a_1^2 + a_2^2 + a_3^2} \\ AB &= \sqrt{(ca_1 - a_1)^2 + (ca_2 - a_2)^2 + (ca_3 - a_3)^2} \\ &= \sqrt{a_1^2(c - 1)^2 + a_2^2(c - 1)^2 + a_3^2(c - 1)^2} \\ &= \sqrt{(c - 1)^2(a_1^2 + a_2^2 + a_3^2)} \\ &= |c - 1|\sqrt{a_1^2 + a_2^2 + a_3^2}. \end{aligned}$$

The rest of the proof follows as in the two-dimensional case, and we conclude that

- B is collinear with A and the origin (using the Triangle Inequality and three cases: $c > 1$, $0 \leq c \leq 1$, and $c < 0$);
- B is $|c|$ times as far from the origin as A is (this is shown in the distance formulas above); and
- if $c > 1$, then A is between O and B ($OA + AB = OB$); if $0 \leq c \leq 1$, then B is between O and A ($OB + AB = OA$); if $c < 0$, then O is between A and B ($OA + OB = AB$).

MORE ON ADDING POINTS

Problem 1 (*Student page 125*) To show that $P = (a_1 + b_1, a_2 + b_2) = A + B$ is a solution to the original problem, we'll simply substitute $x = a_1 + b_1$ and $y = a_2 + b_2$ into the equations:

$$\begin{aligned}(x - a_1)^2 + (y - a_2)^2 &= (a_1 + b_1 - a_1)^2 + (a_2 + b_2 - a_2)^2 \\ &= b_1^2 + b_2^2 \\ (x - b_1)^2 + (y - b_2)^2 &= (a_1 + b_1 - b_1)^2 + (a_2 + b_2 - b_2)^2 \\ &= a_1^2 + a_2^2.\end{aligned}$$

So we have found one solution to the original problem.

Problem 2 (*Student page 126*) In this case, we are checking with specific values for the coordinates. If $A = (3, 4)$ and $B = (12, 5)$, then $A + B = (15, 9)$.

$$\begin{aligned}(x - a_1)^2 + (y - a_2)^2 &= (15 - 3)^2 + (9 - 4)^2 \\ &= 12^2 + 5^2 \\ &= b_1^2 + b_2^2 \\ (x - b_1)^2 + (y - b_2)^2 &= (15 - 12)^2 + (9 - 5)^2 \\ &= 3^2 + 4^2 \\ &= a_1^2 + a_2^2\end{aligned}$$

So $A + B$ is a solution for both equations.

Problem 3 (*Student page 126*) First, let's expand the left side of both equations:

$$\begin{aligned}(x - a_1)^2 + (y - a_2)^2 &= x^2 - 2a_1x + a_1^2 + y^2 - 2a_2y + a_2^2 = b_1^2 + b_2^2 \\ (x - b_1)^2 + (y - b_2)^2 &= x^2 - 2b_1x + b_1^2 + y^2 - 2b_2y + b_2^2 = a_1^2 + a_2^2.\end{aligned}$$

Now we have a system of two equations to solve:

$$\begin{aligned}(1) \quad & x^2 - 2a_1x + y^2 - 2a_2y + a_1^2 + a_2^2 = b_1^2 + b_2^2 \\ (2) \quad & x^2 - 2b_1x + y^2 - 2b_2y + b_1^2 + b_2^2 = a_1^2 + a_2^2.\end{aligned}$$

Subtract equation (1) from equation (2):

$$2a_1x - 2b_1x + 2a_2y - 2b_2y - (a_1^2 + a_2^2) + (b_1^2 + b_2^2) = (a_1^2 + a_2^2) - (b_1^2 + b_2^2).$$

Now factor and move the constants to the right side:

$$2x(a_1 - b_1) + 2y(a_2 - b_2) = 2(a_1^2 + a_2^2) - 2(b_1^2 + b_2^2).$$

Then divide both sides by 2:

$$x(a_1 - b_1) + y(a_2 - b_2) = (a_1^2 + a_2^2) - (b_1^2 + b_2^2).$$

Problem 4 (Student page 127) $A = (3, 4)$ and $B = (2, 5)$, so

$$k = \frac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{(a_1 - b_1)^2 + (a_2 - b_2)^2} = \frac{(9 + 16) - (4 + 25)}{(1)^2 + (-1)^2} = \frac{-4}{2} = -2.$$

Now we will check that $P_1 = A + B = (5, 9)$ and $P_2 = -2(A - B) = (-2, 2)$ both have the property that $PA = OB$ and $PB = OA$:

$$\begin{aligned} P_1A &= \sqrt{(5-3)^2 + (9-4)^2} \\ &= \sqrt{4+25} \\ &= \sqrt{29} \end{aligned}$$

$$\begin{aligned} P_1B &= \sqrt{(5-2)^2 + (9-5)^2} \\ &= \sqrt{9+16} \\ &= 5 \end{aligned}$$

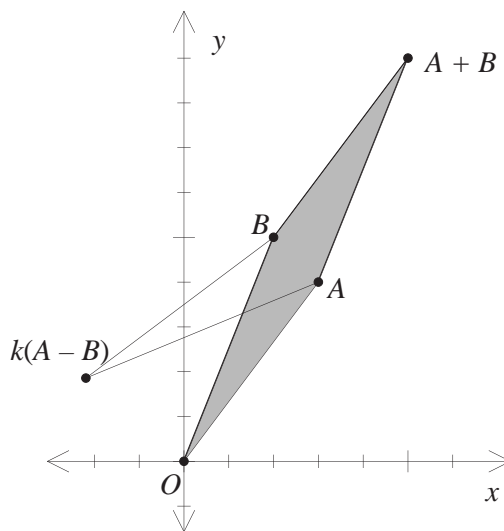
$$\begin{aligned} P_2A &= \sqrt{(-2-3)^2 + (2-4)^2} \\ &= \sqrt{25+4} \\ &= \sqrt{29} \end{aligned}$$

$$\begin{aligned} P_2B &= \sqrt{(-2-2)^2 + (2-5)^2} \\ &= \sqrt{16+9} \\ &= 5 \end{aligned}$$

$$\begin{aligned} OB &= \sqrt{2^2 + 5^2} \\ &= \sqrt{29} \end{aligned}$$

$$\begin{aligned} OA &= \sqrt{3^2 + 4^2} \\ &= 5. \end{aligned}$$

So we see that $P_1A = P_2A = OB$ and $P_1B = P_2B = OA$. In the picture below, we see that $P_1 = A + B$ forms a parallelogram and $P_2 = k(A - B)$ forms a bowtie.



Problem 5 (Student page 127) We would have $k = 0$ only if the numerator is 0. That is, only if

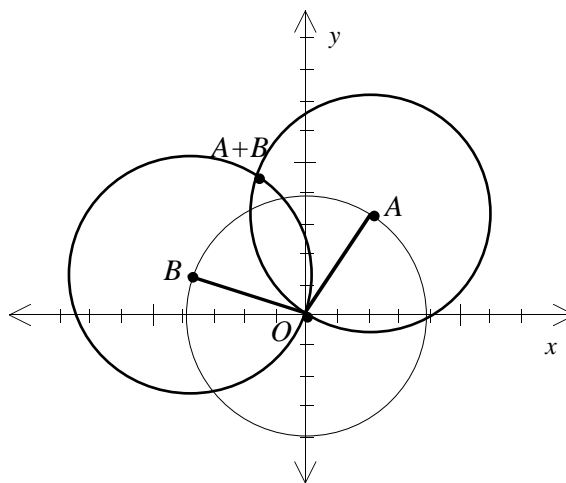
$$\begin{aligned}(a_1^2 + a_2^2) - (b_1^2 + b_2^2) &= 0 \\ a_1^2 + a_2^2 &= b_1^2 + b_2^2.\end{aligned}$$

This says that (a_1, a_2) and (b_1, b_2) are the same distance from the origin. Since $OA = OB$ in the parallelogram, it would be a rhombus.

The bowtie disappears (it collapses into two segments) because P_2 is at the origin; we have only three vertices instead of four. If we go back to the circles at the beginning of the investigation, we see that

- $PA = OB$ puts P on a circle with center A and radius OB .
- $PB = OA$ puts P on a circle with center B and radius OA .

The circle centered at the origin ensures that A and B are the same distance from the origin.



$P_1 = A + B$ and $P_2 = 0$
The “degenerate bowtie” is shown.

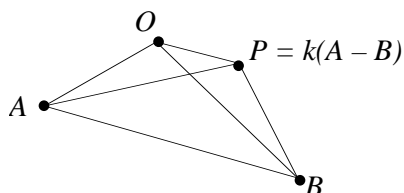
Problem 6 (Student page 127) If we knew that $P_2 = k(A - B)$, then we could substitute into our equation from Problem 3 and solve for k .

$$P_2 = k(a_1 - b_1, a_2 - b_2) = (k(a_1 - b_1), k(a_2 - b_2))$$

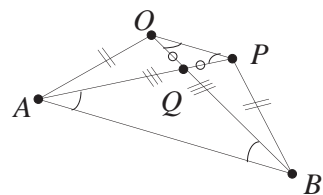
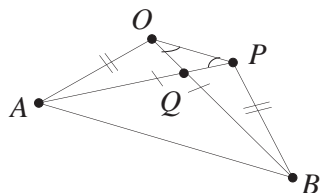
So let $x = k(a_1 - b_1)$ and $y = k(a_2 - b_2)$:

$$\begin{aligned} k(a_1 - b_1)(a_1 - b_1) + k(a_2 - b_2)(a_2 - b_2) &= (a_1^2 + a_2^2) - (b_1^2 + b_2^2) \\ k[(a_1 - b_1)^2 + (a_2 - b_2)^2] &= (a_1^2 + a_2^2) - (b_1^2 + b_2^2) \\ k &= \frac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{(a_1 - b_1)^2 + (a_2 - b_2)^2}. \end{aligned}$$

This is exactly the value of k given in the Student Module.



The solution to Problem 8 depends on using this congruent triangle proof instead of the one given above.



Which of the two proofs is easier for you to follow? Which gives you more insight into “why”? Which requires more knowledge to read or to construct?

Problem 7 (Student page 127) If $OPBA$ is a trapezoid, it is certainly an isosceles trapezoid since we know that $OA = BP$ by construction.

We need to show only that $\overline{AB} \parallel \overline{OP}$.

Since $P = k(A - B)$, we can write $(P - O) = k(A - B)$; from our previous work, we conclude that $\overrightarrow{AB} \parallel \overrightarrow{OP}$. So $OPBA$ is an isosceles trapezoid.

An alternate proof uses congruent and similar triangles and facts about parallel lines. First note that by construction $\overline{OB} \cong \overline{AP}$ and $\overline{OA} \cong \overline{BP}$. Look at triangles OBP and PAO ; they are congruent by SSS (the third side, \overline{OP} , is shared by the two triangles). From this we conclude that $\angle BOP \cong \angle APO$.

Now look at $\triangle QOP$. Its base angles are congruent, so sides \overline{QO} and \overline{QP} are congruent as well.

But these segments are each pieces of larger congruent segments, so the remaining pieces (\overline{QA} and \overline{QB}) are also congruent, and $\triangle QAB$ is also isosceles. In fact, $\triangle QAB \sim \triangle QOP$ because their vertex angles are congruent (vertical angles), so we have SAS similarity.

We can conclude from all of this that $\angle PAB \cong \angle OBA \cong \angle BOP \cong \angle APO$. Looking at \overrightarrow{AB} and \overrightarrow{OP} , we see that alternate interior angles are congruent, so $\overrightarrow{AB} \parallel \overrightarrow{OP}$.

We already know that $\overline{OA} \cong \overline{BP}$ (in fact, we used that fact earlier in our proof), so we can conclude that $OABP$ is an isosceles trapezoid.

Problem 8 (Student page 129) Yes, the discussion does lead to a proof of Theorem 5.10. Here it is:

We came up with the following equations by using the distance formula, letting $PA = OB$ and $PB = OA$ for some point $P = (x, y)$:

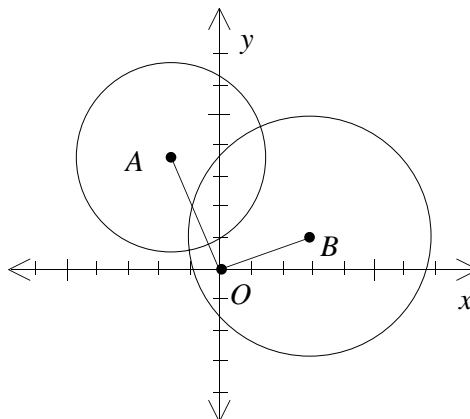
$$\begin{aligned} x^2 - 2a_1x + y^2 - 2a_2y + a_1^2 + a_2^2 &= b_1^2 + b_2^2, \text{ and} \\ x^2 - 2b_1x + y^2 - 2b_2y + b_1^2 + b_2^2 &= a_1^2 + a_2^2. \end{aligned}$$

Some straightforward algebraic manipulations reduced this to just one equation (see the solution for Problem 3):

$$x(a_1 - b_1) + y(a_2 - b_2) = (a_1^2 + a_2^2) - (b_1^2 + b_2^2).$$

Looking at the geometry of the situation, we know that there are at most two solutions

to this equation for (x, y) because we are intersecting two circles.



(Of course, the circles might intersect in only one point or not at all.)

The first solution we found by inspection (and because we knew we wanted that solution): $P_1 = A + B$. To find the second solution, we use Problem 7: If P_2 produces the bowtie, then $OABP$ is an isosceles trapezoid, and $\overline{AB} \parallel \overline{OP}$. From Problem 4, we know that there must be a number k so that $k(A - B) = P_2 - O = P_2$, so $(x, y) = k(a_1 - b_1, a_2 - b_2)$ is the other solution.

Knowing this, we can find the value for k by going back to our equation and replacing $x = k(a_1 - b_1)$ and $y = k(a_2 - b_2)$ and solving for k . We get

$$k = \frac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

Now we have found two solutions: $P_1 = A + B$ and $P_2 = k(A - B)$. We know from the geometry that there are at most two solutions, so we're done.

Problem 9 (Student page 129) In Problem 4, we calculated that for $A = (3, 4)$ and $B = (2, 5)$, $k = -2$ and $P = (-2, 2)$, so

$$s = \frac{-2}{-3} = \frac{2}{3}$$

and

$$sB = \left(\frac{4}{3}, \frac{10}{3}\right).$$

See Problem 3f of Investigation 5.15. Here, we use c rather than k to avoid confusion with the different number k used earlier in this solution.

We know that all multiples of B , including $\frac{2}{3}B$, are on the line \overleftrightarrow{OB} . Because $s < 1$, sB is between O and B and therefore on \overline{OB} .

From earlier work, we know that points along \overline{AP} can be written as $(x, y) = cA + (1 - c)P$ for some c where $0 \leq c \leq 1$. If we can find a c that works for $(x, y) = (\frac{4}{3}, \frac{10}{3})$, we know that sB is on \overline{AP} .

$$c(3, 4) + (1 - c)(-2, 2) = \left(\frac{4}{3}, \frac{10}{3}\right)$$

$$(3c, 4c) + (-2 + 2c, 2 - 2c) = \left(\frac{4}{3}, \frac{10}{3}\right)$$

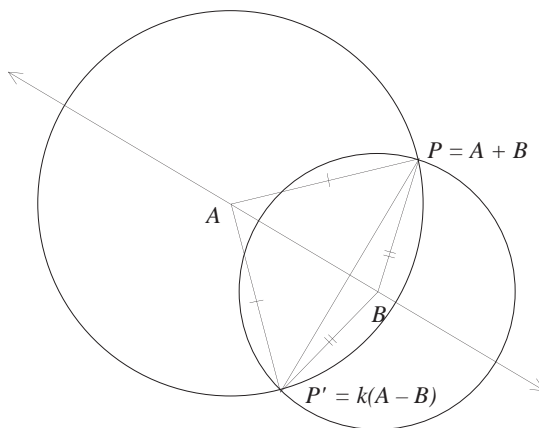
Equating the x -coordinates and y -coordinates, we get

$$\begin{aligned} 3c - 2 + 2c &= \frac{4}{3} & 4c + 2 - 2c &= \frac{10}{3} \\ 5c &= \frac{10}{3} & 2c &= \frac{4}{3} \\ c &= \frac{2}{3} & c &= \frac{2}{3}. \end{aligned}$$

So sB can be written as $\frac{2}{3}A + \frac{1}{3}P$, and it is therefore on \overline{AP} .

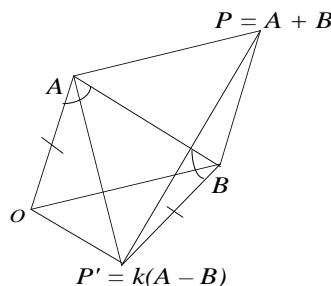
Since it is on both \overline{OB} and \overline{AP} , B must be the point where these two segments intersect.

Problem 10 (Student page 130) First, we show that \overleftrightarrow{AB} is the perpendicular bisector of $\overline{PP'}$. Note that, since both P and P' are on a circle centered at A , we have $\overline{PA} \cong \overline{P'A}$. Also, since P and P' are both on a circle centered at B , $\overline{PB} \cong \overline{P'B}$.



Connecting two points equidistant from the endpoints of segment produces the perpendicular bisector, so \overleftrightarrow{AB} is the perpendicular bisector of $\overline{PP'}$.

Next look at this piece of the picture:



The triangles share side \overline{AB} , and the isosceles trapezoid has congruent sides \overline{OA} and $\overline{P'B}$ and congruent base angles OAB and PBA .

Because $AOP'B$ is an isosceles trapezoid, $\triangle AP'B \cong \triangle BOA$ by SAS. So $\angle P'AB \cong \angle OBA$.

Now look at $\triangle P'AP$. We know that it is isosceles and that \overleftrightarrow{AB} is the perpendicular bisector of the base, so it also bisects the vertex angle. That means $\angle P'AB \cong \angle PAB$. So now we have $\angle PAB \cong \angle OBA$. These are alternate interior angles, so $\overline{OB} \parallel \overline{AP}$.

Next look at $\triangle P'BP$. We know that it is an isosceles triangle and that \overleftrightarrow{AB} is the perpendicular bisector of the base, so it also bisects the vertex angle. That means $\angle P'BA \cong \angle PBA$. So now we have $\angle PBA \cong \angle OAB$. These are alternate interior angles, so $\overline{OA} \parallel \overline{BP}$.

Now we know that $OAPB$ is a parallelogram. Using P' , we have fixed the glitch in Theorem 5.4.

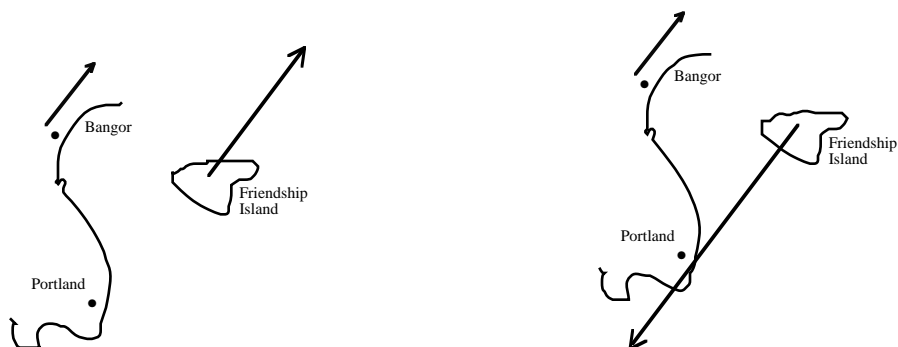
Problem 11 (Student page 130) The glitch in Theorem 5.4 was that we only showed that opposite sides of $OAPB$ were congruent. This proves that $OAPB$ is a parallelogram *if* the figure doesn't cross itself. We could see from particular pictures that $OA(A+B)B$ didn't cross itself in those cases, but we couldn't be sure of *every* case. We fixed the glitch by showing that opposite sides were parallel. This is how it went:

- There are at most two points P where $\overline{OA} \cong \overline{BP}$ and $\overline{OB} \cong \overline{AP}$. One of the points forms a parallelogram with A , B , and O ; the other forms a bowtie.
- We found these two points are $A + B$ and $k(A - B)$ for a specific value of k that depends just on A and B .

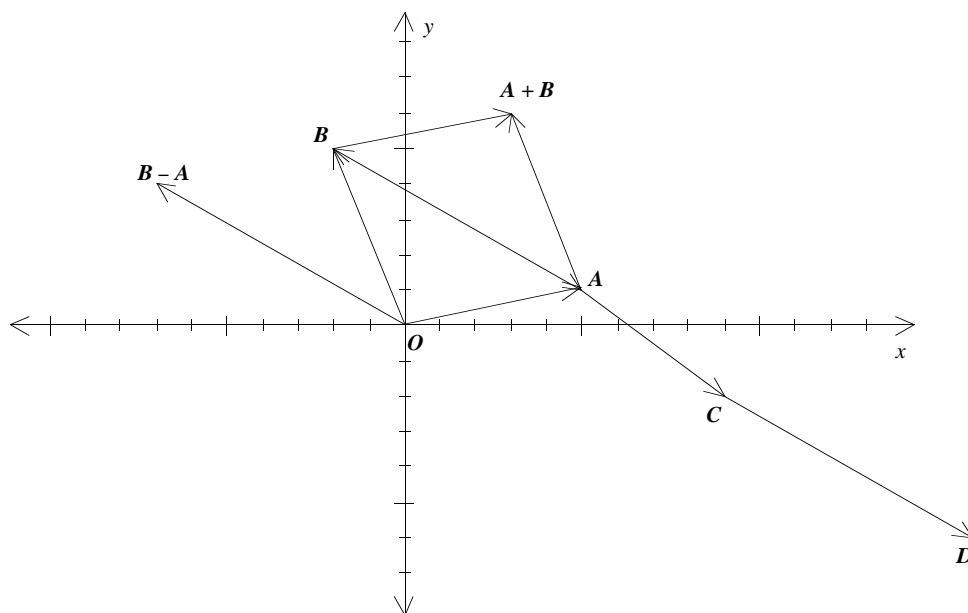
- We showed that if $OAPB$ is a bowtie (with $\overline{OA} \cong \overline{BP}$ and $\overline{OB} \cong \overline{AP}$), then $OABP$ is an isosceles trapezoid.
- We showed that the point $P = k(A - B)$ does produce a bowtie because \overline{OB} and \overline{AP} intersect.
- We used this to show that for $P = A + B$, $OA(A + B)B$ is a parallelogram.

VECTORS AND GEOMETRY

Problem 1 (*Student page 132*) The picture on the left shows a vector with the same direction as the original but twice as long; it represents a wind speed of 20 mph in the same direction as the vector in the Student Module. The picture on the right shows a vector with the opposite direction and four times the magnitude as the original.



Problem 2 (*Student page 133*) This picture shows all of the vectors required:



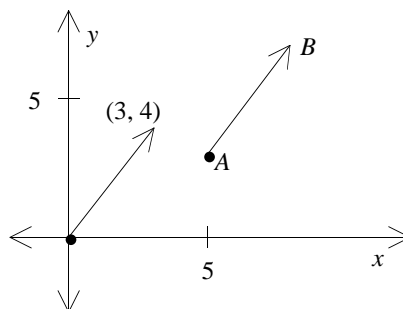
Problem 3 (Student page 133) Same direction:

- \vec{AB} and $\vec{O(B-A)}$
- $\vec{B(A+B)}$ and \vec{OA}
- $\vec{A(A+B)}$ and \vec{OB}
- \vec{BA} and \vec{CD}

Opposite directions:

- \vec{CD} and \vec{AB}
- \vec{CD} and $\vec{O(B-A)}$
- \vec{AB} and \vec{BA}

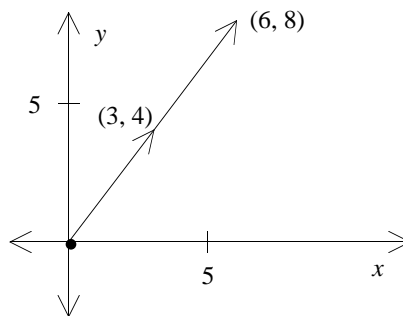
Problem 4 (Student page 133) If two vectors have the same length and direction, they must go “over and up” the same amount. From A to B, you go over (right) 3 and up 4, so from the origin you would land at (3,4).



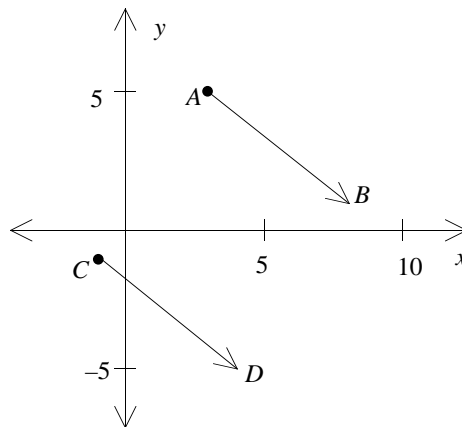
Of course, sometimes the “up” is negative, so you really go down.

Notice that $B - A = (3, 4)$. In general, if you apply HEAD – TAIL to any vector, you get the “over and up” in one neat package.

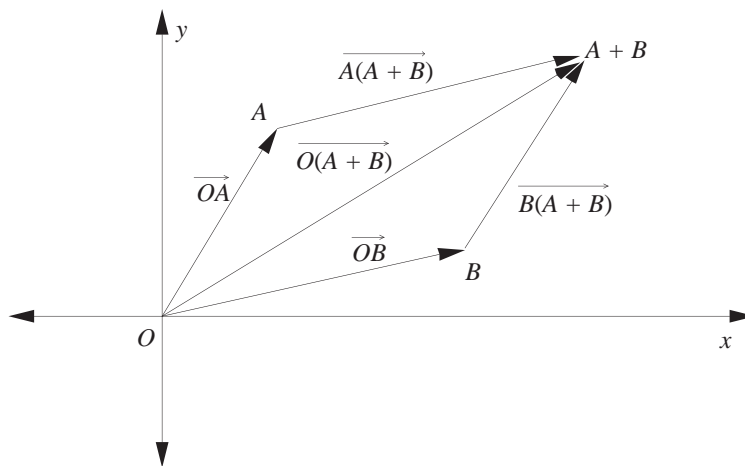
Problem 5 (Student page 133) We can simply double the length of the vector we found in Problem 4 by any of our previous methods. We will land at $(6, 8)$.



Problem 6 (Student page 133) To get from A to B , you go over (right) 5 and down 4. So pick a point C $(-1, -1)$ in this picture and go “over and up” by the same amount. We landed at $D = (4, -5)$.



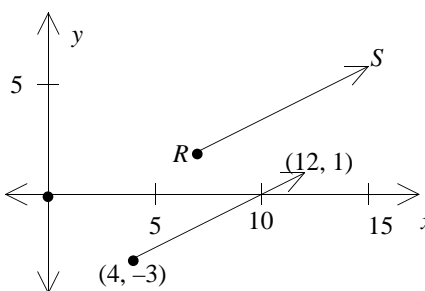
Problem 7 (Student page 133) The diagram should be labeled this way:



Problem 8 (Student page 134) We are given $R = (7, 2)$ and $S = (15, 6)$.

Using the HEAD – TAIL idea from Problem 4,
 $S - R = (8, 4)$. So, starting from $(4, -3)$, we would land at $(4, -3) + (8, 4) = (12, 1)$.

To get from R to S , we go over (right) 8 and up 4. Starting from $(4, -3)$, we would land at $(12, 1)$



$$\begin{aligned} \vec{A' - A} &= \vec{K' - K} = \vec{J' - J} = \\ \vec{L' - L} &= (8, 6) \end{aligned}$$

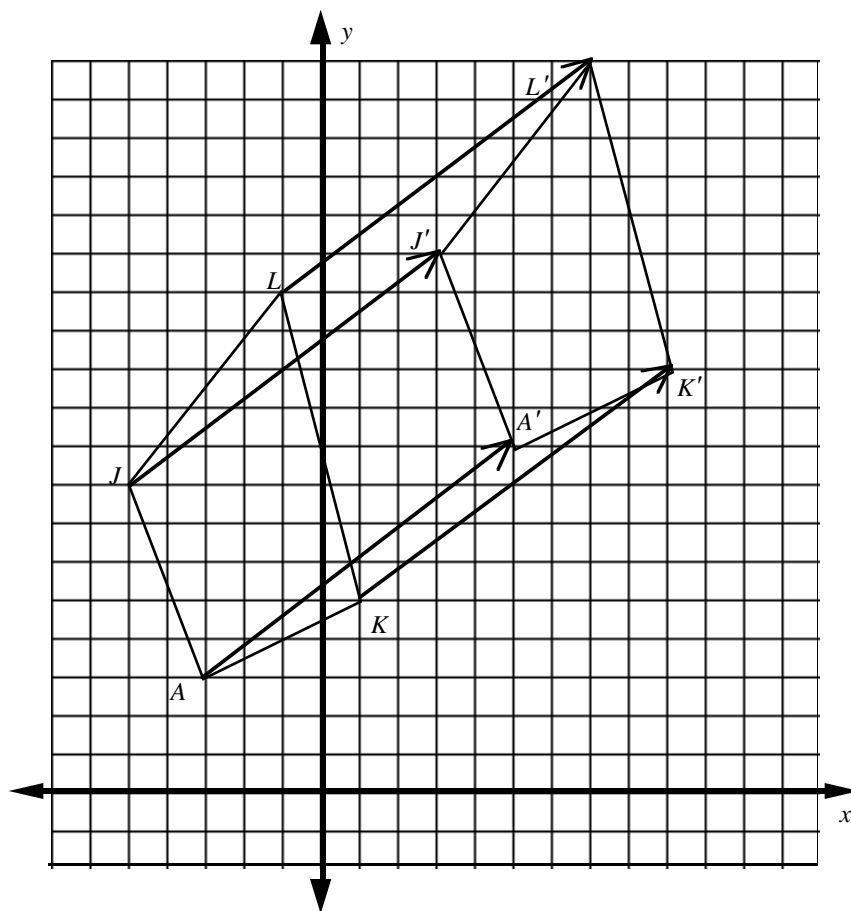
Problem 9 (Student page 134) To convince ourselves that the vectors are all equivalent, we check the “over and up” for $\vec{AA'}$, $\vec{KK'}$, $\vec{JJ'}$, and $\vec{LL'}$:

$\vec{AA'}$: over (right) 8, up 6

$\vec{KK'}$: over (right) 8, up 6

$\vec{JJ'}$: over (right) 8, up 6

$\vec{LL'}$: over (right) 8, up 6.



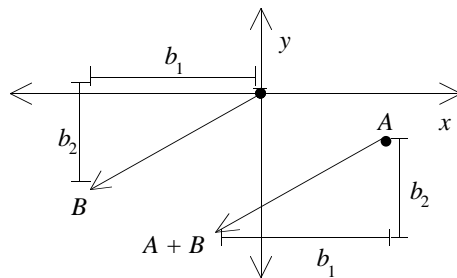
Problem 10 (Student page 135) The statement is true. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$.

Then $A + B = (a_1 + b_1, a_2 + b_2)$.

Assume b_1 and b_2 are positive (the other cases are handled in the same way). The vector from O to B goes over b_1 and up b_2 .

Likewise, the vector from A to $(A + B)$ goes over b_1 and up b_2 .

We see that the two vectors are equivalent.



$(A + B) - A = B - O$, so $\overrightarrow{A(A + B)}$ and \overrightarrow{OB} have the same HEAD – TAIL.

Note: An easy way to check the “over and up” for a vector is to take “HEAD – TAIL.” So here,

$$(\text{HEAD} - \text{TAIL})(\overrightarrow{OB}) = (b_1, b_2) - (0, 0) = (b_1, b_2)$$

$(\text{HEAD} - \text{TAIL})(\overrightarrow{A(A + B)}) = (a_1 + b_1, a_2 + b_2) - (a_1, a_2) = (b_1, b_2)$. Thus, the vectors are equivalent.

Problem 11 (Student page 135) Let’s use the “HEAD – TAIL” trick on $\overrightarrow{AT(A)}$ and $\overrightarrow{BT(B)}$.

Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$.

$$(\text{HEAD} - \text{TAIL})(\overrightarrow{AT(A)}) = (a_1 + 10, a_2 + 5) - (a_1, a_2) = (10, 5)$$

$$(\text{HEAD} - \text{TAIL})(\overrightarrow{BT(B)}) = (b_1 + 10, b_2 + 5) - (b_1, b_2) = (10, 5),$$

$$T(A) = A + (10, 5)$$

$$T(B) = B + (10, 5)$$

$$T(A) - A =$$

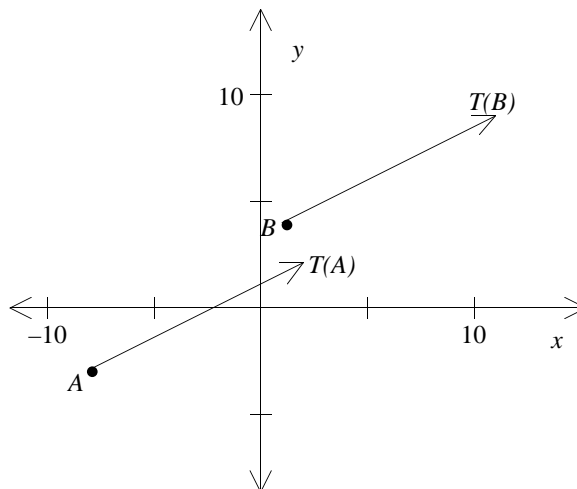
$$(A + (10, 5)) - A = (10, 5)$$

$$\text{and } T(B) - B =$$

$$(B + (10, 5)) - B = (10, 5).$$

So, HEAD – TAIL is the same for both.

so $\overrightarrow{AT(A)}$ and $\overrightarrow{BT(B)}$ are equivalent.



Problem 12 (Student page 135) For each vector $\overrightarrow{XT(X)}$,

$$\text{HEAD} - \text{TAIL} = (x_1 + 8, x_2 + 5) - (x_1, x_2) = (8, 5).$$

Another way to say this without coordinates is

$$\text{HEAD} - \text{TAIL} = (X + (8, 5)) - X = (8, 5).$$

So all of the vectors are equivalent.

Problem 13 (Student page 135) If $A = (a_1, a_2)$ and $B = (b_1, b_2)$ then

$$(\text{HEAD} - \text{TAIL})(\overrightarrow{AB}) = (b_1 - a_1, b_2 - a_2).$$

We want a vector C so that

$$(\text{HEAD} - \text{TAIL})(\overrightarrow{OC}) = (c_1 - 0, c_2 - 0) = (b_1 - a_1, b_2 - a_2).$$

$$\text{Thus, } C = (b_1 - a_1, b_2 - a_2).$$

This result is often stated like this: If you move \overrightarrow{OC} to the origin, you get $\overrightarrow{O(B-A)}$.

A “coordinate-free” way to do this is to say that we want a point C so that

$$C - O = B - A.$$

But $C - O = C$, so $C = B - A$.

Problem 14 (Student page 136) Two vectors, \overrightarrow{AB} and \overrightarrow{CD} , are equivalent if

$$(\text{HEAD} - \text{TAIL})(\overrightarrow{AB}) = (\text{HEAD} - \text{TAIL})(\overrightarrow{CD}).$$

That is, if $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, and $D = (d_1, d_2)$, then \overrightarrow{AB} and \overrightarrow{CD} are equivalent if

$$(b_1 - a_1, b_2 - a_2) = (d_1 - c_1, d_2 - c_2),$$

or, put another way, \overrightarrow{AB} and \overrightarrow{CD} are equivalent if

$$B - A = D - C.$$

Problem 15 (Student page 136) Using our HEAD – TAIL rule:

$$A = (a_1, a_2), B = (b_1, b_2), B - A = (b_1 - a_1, b_2 - a_2)$$

$$(\text{HEAD} - \text{TAIL})(\overrightarrow{AB}) = (b_1 - a_1, b_2 - a_2)$$

$$(\text{HEAD} - \text{TAIL})(\overrightarrow{O(B-A)}) = (b_1 - a_1, b_2 - a_2).$$

So \overrightarrow{AB} is equivalent to $\overrightarrow{O(B-A)}$.

Things like $(B - A) - O = B - A$ follow from the “algebra of points” discussed in Investigation 5.15.

Put another way,

$$(\text{HEAD} - \text{TAIL})(\overrightarrow{AB}) = B - A,$$

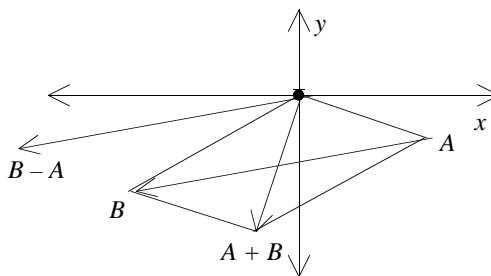
and

$$(\text{HEAD} - \text{TAIL})(\overrightarrow{O(B-A)}) = (B - A) - O.$$

Since $(B - A) - O = B - A$, \overrightarrow{AB} is equivalent to $\overrightarrow{O(B-A)}$.

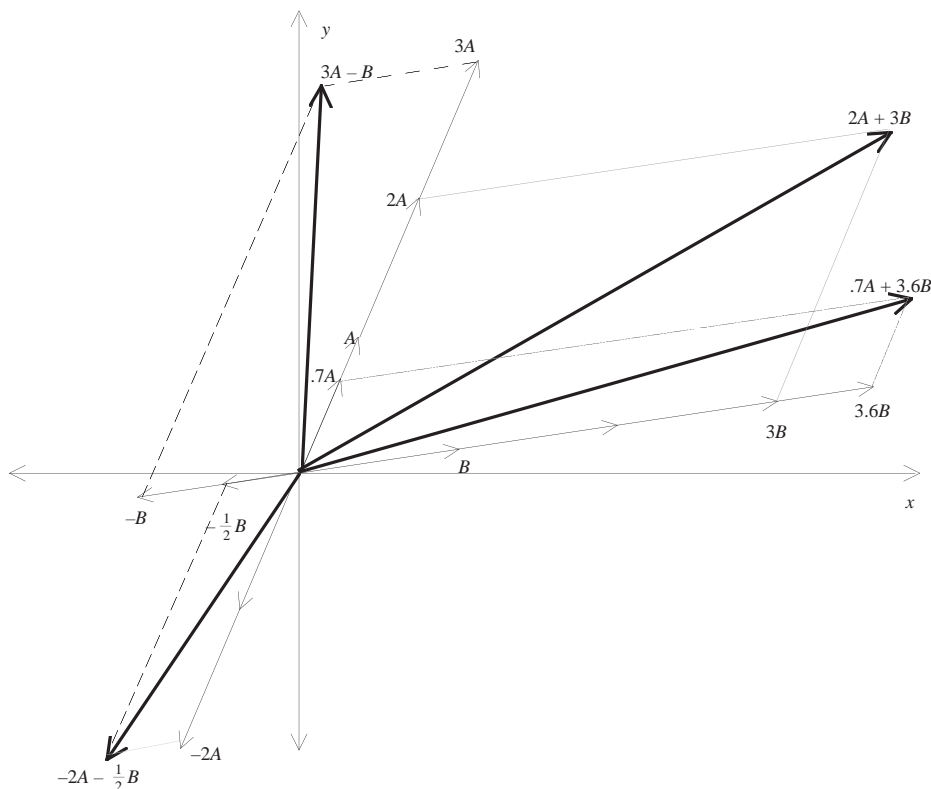
This is exactly what we showed in Theorem 5.12 (proved in Problem 16). Here we are just noticing that \vec{AB} is the other diagonal of the parallelogram whose vertices are O , A , $(A + B)$, and B .

Problem 16 (Student page 139) The other diagonal of the parallelogram is \vec{AB} .



But by Problem 15, \vec{AB} is equivalent to $\vec{O(B - A)}$.

Problem 17 (Student page 139)



Problem 18 (Student page 140)

- a. $x = 1, y = 1$
- b. $x = 3, y = 1$
- c. $x = 1, y = 10$
- d. $x = 6, y = -1$
- e. $x = 8, y = -8$
- f. $x = -10, y = 11$

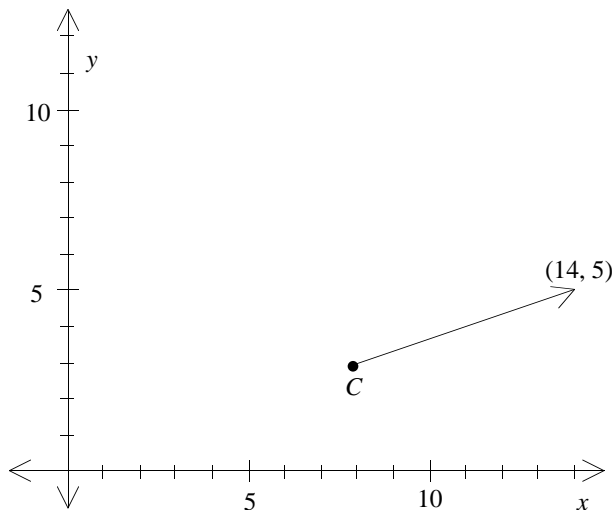
In general, $(a, b) = b(1, 1) + (a - b)(1, 0)$.

Problem 19 (Student page 141)

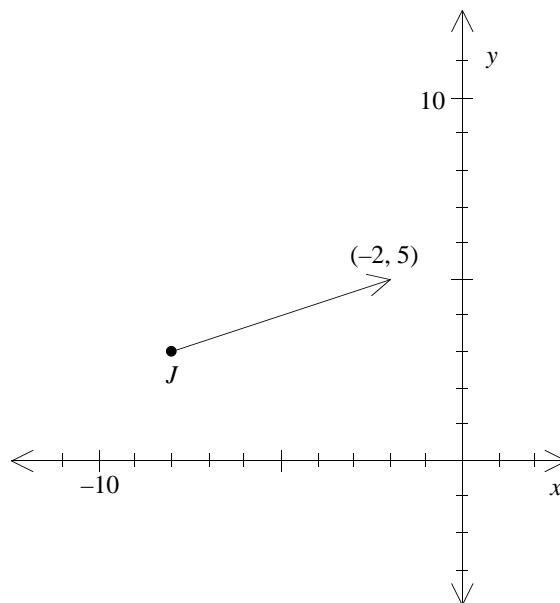
- a. (HEAD – TAIL)(\vec{AB}) = $(6, 2)$
- b. $P + (6, 2) = (8, -1)$

Problem 20 (Student page 142) (HEAD – TAIL)(\vec{AB}) = $B - A = (6, 2)$

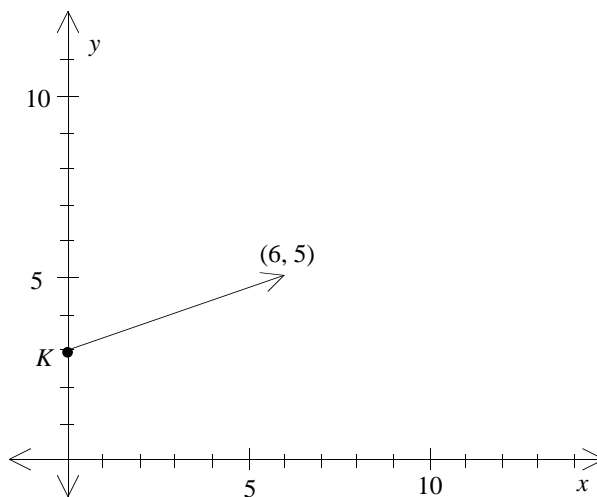
- a. $C = (8, 3), C + (6, 2) = (14, 5)$



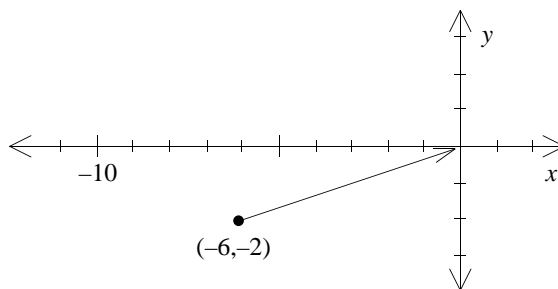
b. $J = (-8, 3)$, $J + (6, 2) = (-2, 5)$



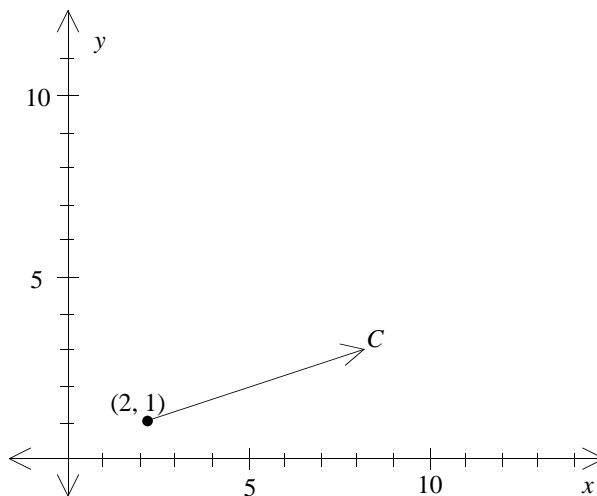
c. $K = (0, 3)$, $K + (6, 2) = (6, 5)$



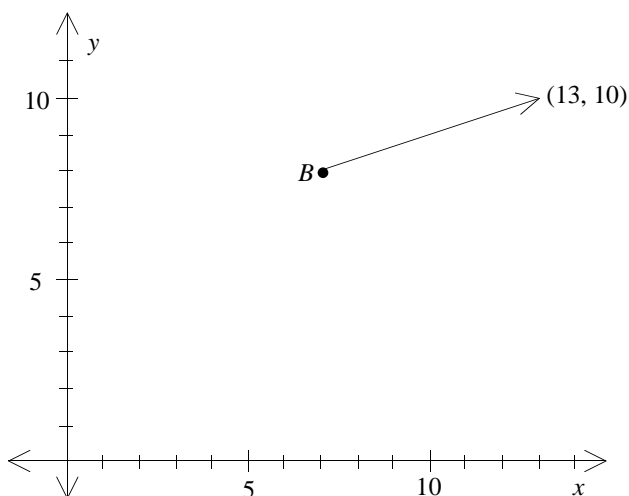
d. $P + (6, 2) = (0, 0)$, $P = (-6, -2)$



e. $P + (6, 2) = (8, 3)$, $P = (2, 1)$



f. $B = (7, 8)$, $B + (6, 2) = (13, 10)$



Problem 21 (Student page 143) We are given $A = (3, 4)$, $B = (9, 0)$, $C = (-1, 2)$, and $D = (5, -2)$.

Moving \overrightarrow{AB} to the origin gives $(9, 0) - (3, 4) = (6, -4)$.

Moving \overrightarrow{CD} to the origin gives $(5, -2) - (-1, 2) = (6, -4)$.

What shape is $ABCD$?

So \overrightarrow{AB} is equivalent to \overrightarrow{CD} .

Moving \overrightarrow{AC} to the origin gives $(-1, 2) - (3, 4) = (-4, -2)$.

Moving \overrightarrow{BD} to the origin gives $(5, -2) - (9, 0) = (-4, -2)$.

So \overrightarrow{AC} is equivalent to \overrightarrow{BD} .

Problem 22 (Student page 143) In the original quadrilateral,

$A = (-1, 9)$, $B = (2, 12)$, $C = (8, 10)$, and $D = (7, 7)$.

So, in the first picture, the new vertices are

$$\begin{aligned} A - A &= (0, 0), & C - A &= (9, 1) \\ B - A &= (3, 3), & D - A &= (8, -2). \end{aligned}$$

In the second picture, the new vertices are

$$\begin{aligned} A - B &= (-3, -3), & C - B &= (6, -2) \\ B - B &= (0, 0), & D - B &= (5, -5). \end{aligned}$$

And in the third picture, the new vertices are

$$\begin{aligned} A - C &= (-9, -1), & C - C &= (0, 0) \\ B - C &= (-6, 2), & D - C &= (-1, -3). \end{aligned}$$

In the fourth picture, the new vertices are

$$\begin{aligned} A - D &= (-8, 2), & C - D &= (1, 3) \\ B - D &= (-5, 5), & D - D &= (0, 0). \end{aligned}$$

Problem 23 (Student page 145) We will use the HEAD – TAIL test for equivalent vectors.

Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $X = (x_1, x_2)$.

We want to show $(\text{HEAD} - \text{TAIL})(\overrightarrow{AB}) = (\text{HEAD} - \text{TAIL})(\overrightarrow{A+X})(\overrightarrow{B+X})$.

$$\begin{aligned} (\text{HEAD} - \text{TAIL})(\overrightarrow{AB}) &= (b_1 - a_1, b_2 - a_2) \\ (\text{HEAD} - \text{TAIL})(\overrightarrow{A+X})(\overrightarrow{B+X}) &= ((b_1 + x_1) - (a_1 + x_1), (b_2 + x_2) - (a_2 + x_2)) \\ &= (b_1 - a_1, b_2 - a_2) \end{aligned}$$

Using the algebra of points developed in Investigation 5.15, we can do this in a coordinate-free way:

$$(B + X) - (A + X) = B + X - A - X = B - A.$$

So adding the same point to the head and the tail of a vector produces an equivalent vector.

Problem 24 (Student page 146) First, calculate HEAD – TAIL for each of the vectors; then use those calculations to answer the questions.

$$\begin{aligned} B - A &= (6, 4) & D - C &= (1, -7) & F - E &= (6, 4) \\ G - H &= (-8, 4) & L - J &= (-6, -4) & M - N &= (5, 5) \\ K - I &= (-9, 4) & Q - P &= (-4, 2) & R - S &= (6, -3) \end{aligned}$$

- a. Parallel vectors have the same heading, so when moved to the origin will be on the same line. Look for the differences that are multiples of each other.

$$B - A = F - E = -1(L - J), \text{ so } \vec{AB} \parallel \vec{EF} \parallel \vec{JL}.$$

$$G - H = 2(Q - P) = -\frac{4}{3}(R - S), \text{ so } \vec{HG} \parallel \vec{PQ} \parallel \vec{SR}.$$

- b. By checking visually, we can see which of the parallel vectors go in the same direction:

$$\vec{AB} \text{ and } \vec{EF}, \vec{HG} \text{ and } \vec{PQ}.$$

- c. Likewise, we can see which of the parallel vectors go in opposite directions:

$$\vec{AB} \text{ and } \vec{JL}, \vec{EF} \text{ and } \vec{JL}, \vec{HG} \text{ and } \vec{SR}, \vec{PQ} \text{ and } \vec{SR}.$$

- d. Equivalent vectors will be the same when moved to the origin. There is only one pair: \vec{AB} is equivalent to \vec{EF} .

- e. If two vectors, say \vec{AB} and \vec{CD} , are parallel in the same direction, then

$$(\text{HEAD} - \text{TAIL})(\vec{AB}) = k \cdot (\text{HEAD} - \text{TAIL})(\vec{CD}) \text{ where } k > 0.$$

If two vectors, say \vec{EF} and \vec{GH} , are parallel in opposite directions, then

$$(\text{HEAD} - \text{TAIL})(\vec{EF}) = k \cdot (\text{HEAD} - \text{TAIL})(\vec{GH}) \text{ where } k < 0.$$

Problem 25 (Student page 149) We calculate $B - A = (4, -7)$ and $D - C = (-4, 7)$.

$B - A$ is not equal to $D - C$, so the vectors are not equivalent. Notice, though, that $B - A = -1(D - C)$, so the vectors are parallel in the opposite direction. And, in fact, \vec{AB} is equivalent to \vec{DC} .

Problem 26 (Student page 149) We calculate that $B - A = (4, -7)$ and $D - C = (8, -14)$.

Can k ever equal 0?

Notice $B - A = \frac{1}{2}(D - C)$, so \overrightarrow{AB} and \overrightarrow{CD} are parallel in the same direction.

Problem 27 (Student page 149) We can calculate with coordinates:

Tail of \overrightarrow{AB} :

$$\begin{aligned} & 2((3, 5) - (3, 5)) + (9, 1) \\ &= 2(0, 0) + (9, 1) \\ &= (9, 1) = C \end{aligned}$$

Head of \overrightarrow{AB} :

$$\begin{aligned} & 2((7, -2) - (3, 5)) + (9, 1) \\ &= 2(4, -7) + (9, 1) \\ &= (17, -13) = D \end{aligned}$$

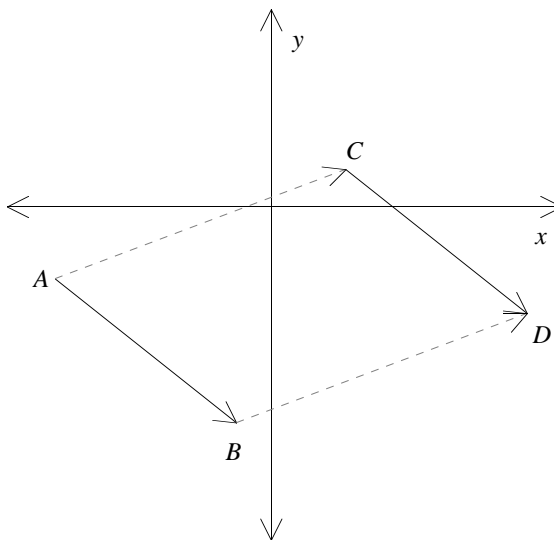
So you get the vector from C to D .

We could also reason this way:

$X - A$ moves \overrightarrow{AB} to the origin. $2(X - A)$ stretches that vector at the origin by 2. (It's now equivalent to \overrightarrow{CD} .)

$2(X - A) + C$ moves the vector so the tail is at C . We get \overrightarrow{CD} .

Problem 28 (Student page 149) In the picture, \vec{AB} is equivalent to \vec{CD} . It also appears that \vec{AC} is equivalent to \vec{BD} . How can we be sure? Here are three ways to do it:



- One pair of opposite sides that are both congruent and parallel guarantees that we have a parallelogram, so $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$ (since the vectors are equivalent).

Then $ABDC$ is a parallelogram, so $\overline{AC} \cong \overline{BD}$ and $\overline{AC} \parallel \overline{BD}$.

So \vec{AC} and \vec{BD} are equivalent.

- We can check with “algebra of points”: $B - A = D - C$ since \vec{AB} and \vec{CD} are equivalent. We can add C to both sides (translating vectors from the origin to C):

$$(B - A) + C = (D - C) + C$$

$$B + (C - A) = D.$$

Now we can add $-B$ to both sides (again, translating vectors):

$$-B + B + (C - A) = D - B$$

$$C - A = D - B.$$

So, by our HEAD – TAIL test, \vec{AC} is equivalent to \vec{BD} .

- We can check with coordinates:

$$B - A = (b_1 - a_1, b_2 - a_2)$$

$$D - C = (d_1 - c_1, d_2 - c_2)$$

Since \overrightarrow{AB} is equivalent to \overrightarrow{CD} ,

$$b_1 - a_1 = d_1 - c_1 \text{ and } b_2 - a_2 = d_2 - c_2.$$

Using algebra, we can rearrange the terms of these equations to get

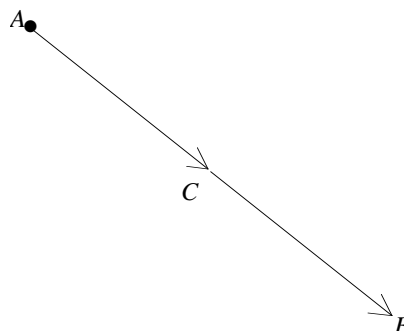
$$c_1 - a_1 = d_1 - b_1 \text{ and } c_2 - a_2 = d_2 - b_2.$$

So $C - A = D - B$.

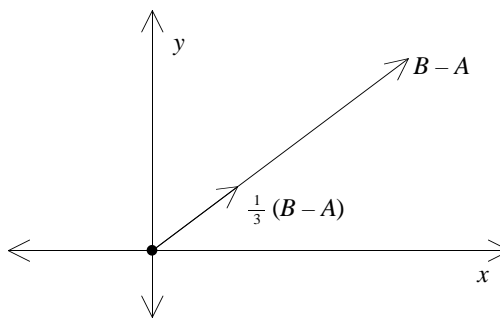
Thus, \overrightarrow{AC} is equivalent to \overrightarrow{BD} .

USING VECTORS TO SOLVE PROBLEMS

Problem 1 (Student page 151) Let $C = \frac{1}{2}(A + B)$. Since $C - A = B - C = \frac{1}{2}(B - A)$, \vec{AC} is equivalent to \vec{CB} . Thus, C is the midpoint of \overline{AB} .



Problem 2 (Student page 151) Think of \overline{AB} as a vector with tail at A and head at B . Translate it to the origin so that its tail is O , and its head is $B - A$. The point $\frac{1}{3}$ of the way from O to $B - A$ is $\frac{1}{3}(B - A)$.



Now translate the vector back to where it began by adding A . The point $\frac{1}{3}(B - A)$ moves to

$$\frac{1}{3}(B - A) + A = \frac{2}{3}A + \frac{1}{3}B.$$

Problem 3 (Student page 151) Using the same reasoning as in the previous problem gives

$$\frac{1}{4}(B - A) + A = \frac{3}{4}A + \frac{1}{4}B.$$

Suppose k is a number between 0 and 1. What point on \overline{AB} is a “ k th” of the way between A and B ?

Problem 4 (Student page 151) Let $C = (c_1, c_2)$. For B to be the midpoint of \overline{AC} , $B - A$ must equal $C - B$, so

$$(4, -4) = (c_1 - 7, c_2 - 1),$$

and $C = (11, -3)$.

Another way to think about it:

If

$$B = \frac{1}{2}(A + C),$$

then

$$2B = A + C,$$

so

$$\begin{aligned} C &= 2B - A \\ &= 2(7, 1) - (3, 5) \\ &= (14, 2) - (3, 5) = (11, -3). \end{aligned}$$

Problem 5 (Student page 151) The midpoints of quadrilateral $ABCD$ are $E = (\frac{3}{2}, 10)$, $F = (\frac{15}{2}, \frac{11}{2})$, $G = (9, 2)$, and $H = (3, \frac{13}{2})$. Think of \overrightarrow{EF} and \overrightarrow{HG} as vectors.

Some arithmetic shows that

$$F - E = G - H.$$

Thus, the two vectors are equivalent, implying that \overrightarrow{EF} and \overrightarrow{HG} are equal in length and parallel. Similar reasoning shows that \overrightarrow{EH} and \overrightarrow{FG} are equal in length and parallel. Thus, $EFGH$ is a parallelogram.

Problem 6 (Student page 151)

- The midpoints are $\frac{A+B}{2}$, $\frac{B+C}{2}$, $\frac{C+D}{2}$, and $\frac{A+D}{2}$.
- The side whose endpoints are $\frac{A+B}{2}$ and $\frac{B+C}{2}$ is parallel (and equal in length) to the side with endpoints $\frac{A+D}{2}$ and $\frac{C+D}{2}$ since

$$\frac{B+C}{2} - \frac{A+B}{2} = \frac{C+D}{2} - \frac{A+D}{2} = \frac{C-A}{2}.$$

Similar reasoning shows that the other two sides of the quadrilateral are also parallel. Thus, the quadrilateral is a parallelogram.

Problem 7 (Student page 151)

- a. Since A , B , C , and D are the midpoints of \overline{PX} , \overline{XY} , \overline{YZ} , and \overline{ZQ} , we can write

$$P + X = 2A \quad (\text{i})$$

$$X + Y = 2B \quad (\text{ii})$$

$$Y + Z = 2C \quad (\text{iii})$$

$$Z + Q = 2D. \quad (\text{iv})$$

We're interested in \overrightarrow{QP} . Thus it would be nice to use the four equations above to find an expression for $P - Q$. Let's see if we can eliminate X , Y , and Z :

Subtracting equation (ii) from equation (i) gives $P - Y = 2A - 2B$. Adding this equation to equation (iii) gives $P + Z = 2A - 2B + 2C$. Subtracting equation (iv) from this equation gives

$$P - Q = 2A - 2B + 2C - 2D.$$

Thus, the vector \overrightarrow{QP} depends entirely upon the vertices of quadrilateral $ABCD$, which are fixed. Thus, for every location of point P , the vector \overrightarrow{QP} will have the same length and direction. Pretty neat, isn't it?

Here is another way to transform the equations:

Solve equation (i) for X :

$$X = 2A - P.$$

Solve equation (ii) for Y , and then replace X with $2A - P$:

$$Y = 2B - X = 2B - (2A - P) = 2B - 2A + P.$$

Do you see a pattern here?

Solve equation (iii) for Z , and then replace Y with $2B - 2A + P$:

$$Z = 2C - Y = 2C - (2B - 2A + P) = 2C - 2B + 2A - P.$$

Solve equation (iv) for Q , and then replace Z with $2C - 2B + 2A - P$:

$$Q = 2D - Z = 2D - (2C - 2B + 2A - P) = 2D - 2C + 2B - 2A + P.$$

Then,

$$\begin{aligned} P - Q &= P - (2D - 2C + 2B - 2A + P) \\ &= P - 2D + 2C - 2B + 2A - P \\ &= 2A - 2B + 2C - 2D, \end{aligned}$$

as before.

- b. When $P - Q = 0$,

$$2A - 2B + 2C - 2D = 0.$$

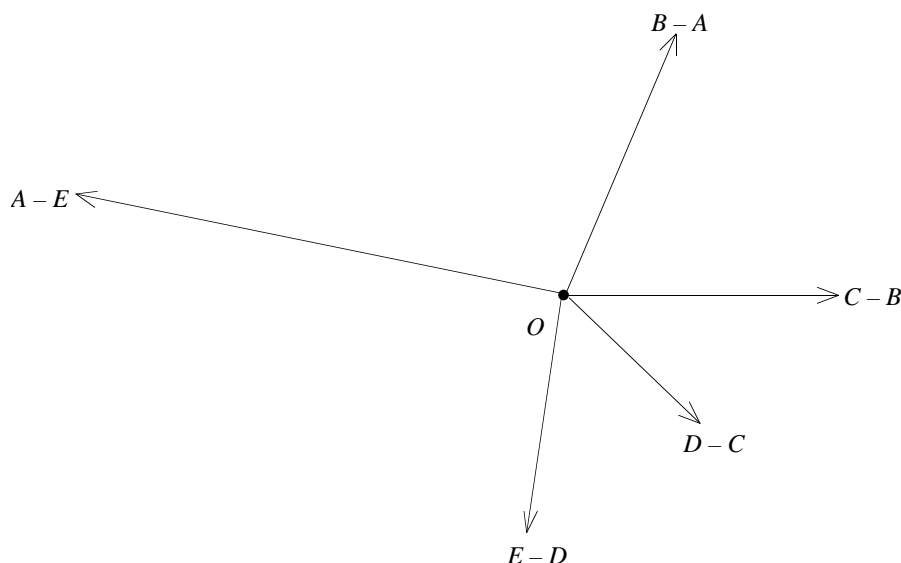
We can write this as $A - B = D - C$ (showing that sides \overline{AB} and \overline{DC} are parallel) or $A - D = B - C$ (showing that sides \overline{AD} and \overline{BC} are parallel).

Thus, for P and Q to be the same point, $ABCD$ must be a parallelogram.

For another solution to this problem, see the section “Midpoints in Quadrilaterals” in Investigation 4.12 in the *Connected Geometry* module *A Matter of Scale*.

Problem 8 (Student page 152)

a.



b.

$$(A - E) + (B - A) + (C - B) + (D - C) + (E - D) = 0$$

This is called a “collapsing sum.” Why do you think that is?

Problem 9 (Student page 153) To find the midpoint of a vector, add the head and tail and divide by 2. The two diagonals of the parallelogram are $\overrightarrow{O(A+B)}$ and \overrightarrow{AB} . The midpoints of both diagonals are the same: $\frac{A+B}{2}$. Thus, the diagonals bisect each other. “Bisect each other” means to cut each other in half, so two segments bisect each other if they have the same midpoint.

Problem 10 (Student page 154) We have

$$t^2 = r^2 = a_1^2 + a_2^2$$

$$u^2 = s^2 = b_1^2 + b_2^2$$

$$x^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

$$y^2 = (a_1 + b_1)^2 + (a_2 + b_2)^2.$$

Algebra now shows that $x^2 + y^2 = r^2 + s^2 + t^2 + u^2$.

Problem 11 (Student page 154) In Problem 2, we saw that the point $\frac{2}{3}A + \frac{1}{3}B$ is on \overline{AB} , $\frac{1}{3}$ of the way from A to B . In Problem 3, we saw that the point $\frac{3}{4}A + \frac{1}{4}B$ is $\frac{1}{4}$ of the way from A to B . Let's show that in general, the point $kA + (1-k)B$ is $(1-k)$ of the way from A to B (where $0 \leq k \leq 1$).

Think of \overrightarrow{AB} as a vector. Translate it to the origin so that its head is now at $B - A$. The point $(1-k)$ of the way from O to $B - A$ is $(1-k)(B - A)$. Now translate the vector back to where it began. The point $(1-k)(B - A)$ moves to

$$(1-k)(B - A) + A = kA + (1-k)B.$$

Problem 12 (Student page 154) Let's show that P is $\frac{2}{3}$ of the way from B to the midpoint of \overline{AC} . The midpoint of \overline{AC} is $\frac{A+C}{2}$. From Problem 11, we know that the point $\frac{2}{3}$ of the way from B to $\frac{A+C}{2}$ is

$$\frac{1}{3}B + \frac{2}{3}\left(\frac{A+C}{2}\right).$$

Simplifying the expression gives $\frac{1}{3}(A + B + C)$.

We can use a similar argument to prove the statement for the other two vertices, A and C .

Look what you have proved here. If $P = \frac{1}{3}(A + B + C)$, then P is on each median of $\triangle ABC$, $\frac{2}{3}$ of the way from any vertex to the midpoint of the opposite side. This is a famous result:

THEOREM *Concurrence of Medians*

The medians of a triangle are concurrent at a point that is $\frac{2}{3}$ of the way from any vertex to the midpoint of the opposite side.

Three or more points are **concurrent** if they all meet at a single point.

The point at which the medians all meet is called the *centroid* of the triangle, and it has many interesting properties. For example,

- Archimedes (c. 287 B.C.–212 B.C.) discovered (and proved) that the centroid is the “center of gravity” for a thin triangular plate of uniform density: if you take such a plate, it will balance on a pencil point if you put the point at the centroid of the triangle. Put another way, if you have three equal weights on a uniform board at A , B , and C (and no other weights anywhere), the board will balance at the centroid of $\triangle ABC$.
- Suppose A , B , and C are points on the plane. For any point P , you can calculate the sum of the squares of the distances to A , B , and C :

$$PA^2 + PB^2 + PC^2.$$

This sum is smallest if P is the centroid of $\triangle ABC$.

- If P is the centroid of $\triangle ABC$, then P “substitutes” for A , B , and C in the sense that

$$A + B + C = P + P + P.$$

On the other hand, if you want to minimize $PA + PB + PC$, you need another point called the **Fermat point**. See the *Connected Geometry* module *Optimization* for more details.

Problem 13 (Student page 154) A reasonable definition for the “center of population” is to imagine the cities as weights at three points on a uniform board. If we think of each person (in each city) as a weight of equal size, all of these forces balance at the point George describes.

Problem 14 (Student page 155) Scaling the coordinates amounts to weighting the three cities by their respective populations. Martha’s point has the property that it would balance a triangle with these weights at the vertices.

Another way to think about it is that Martha's point can "substitute" for the three cities. Suppose, for example, that the three cities located at A , B , and C had populations of 100,000, 120,000, and 80,000. Then Martha's point is

$$\begin{aligned} P &= \frac{100,000}{100,000 + 120,000 + 80,000}A + \frac{120,000}{100,000 + 120,000 + 80,000}B \\ &\quad + \frac{80,000}{100,000 + 120,000 + 80,000}C \\ &= \frac{100,000}{300,000}A + \frac{120,000}{300,000}B + \frac{80,000}{300,000}C \end{aligned}$$

Then

$$\frac{100,000}{300,000}A + \frac{120,000}{300,000}B + \frac{80,000}{300,000}C = \frac{100,000}{300,000}P + \frac{120,000}{300,000}P + \frac{80,000}{300,000}P.$$

To see this, note that the right-hand side simplifies to P , so the whole thing comes from the definition.

Can you prove this if 100,000, 120,000, and 80,000 are *any* three numbers?

Problem 15 (Student page 155) Let's start by calling O the origin. F , the midpoint of \overline{OD} , is $\frac{D}{2}$. Since E is the midpoint of \overline{DA} , it is equal to $\frac{D+A}{2}$. For $DEBF$ to be a parallelogram, $B - E$ must equal $F - D$. Thus, $B - \frac{D+A}{2} = \frac{D}{2} - D$, so $B = \frac{A}{2}$.

What do we know? For every location of point A , point B will lie along \overline{OA} at $\frac{A}{2}$. Thus, the figure traced by point B will be a dilated, half-size copy of the figure traced by A .